

LECTURE NOTE

SUBJECT - ENGG. MATHEMATICS III

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COMPLEX NUMBER

⇒ Let's consider a quadratic equation

$$ax^2 + bx + c = 0, \quad a, b, c \in \mathbb{R}$$

e.g. $x^2 + 1 = 0, \quad x \in \mathbb{R}$

$$\Rightarrow x^2 = -1$$

$$\Rightarrow x = \sqrt{-1} \quad (\text{so there is no solution})$$

So, let $\sqrt{-1} = i$ (i.e. imaginary no. [does not exist])
and value of $i^2 = -1$

Example of imaginary no.s $\rightarrow 3i, -4i, \sqrt{3}i, \frac{i}{2}, \text{etc.}$

i.e. imaginary numbers are of the form $i\mathbb{R}$.

⇒ Now if we form numbers like $\mathbb{R} + i\mathbb{R}$

↓

it is called complex Number.

having both real part and imaginary part.

Basically complex Numbers are defined as

$$\mathbb{C} = \{ z \mid z = x + iy, \text{ where } x, y \in \mathbb{R} \}$$

In complex Numbers $Z = x + iy, \quad x, y \in \mathbb{R}$

where $x = \text{Real part of } Z$

$y = \text{Imaginary part of } Z$

Forc example

$$(i) z = 3 - 4i$$

$$(ii) z = -2 + 5i$$

$$(iii) z = \sqrt{2} + i$$

$$(iv) z = 1 + 3i$$

Now $z = 3i$ it is also a complex number where real part is zero.

~~***~~ And here $z = 3i$ is said to be purely imaginary as $\text{Re}(z) = 0$

Again $z = -7$ is also a complex Number.

where imaginary part is zero.

~~***~~ So $z = -7$ is said to be purely real numbers.

Now

$$\Rightarrow \text{we have } i = \sqrt{-1}$$
$$\boxed{i^2 = -1} \quad \underline{\hspace{2cm}} \quad (*)$$

$$i^3 = i^2 \cdot i$$
$$= -1 \cdot i$$

$$\Rightarrow \boxed{i^3 = -i} \quad \underline{\hspace{2cm}} \quad (*)$$

$$i^4 = i^2 \cdot i^2$$
$$= (-1) \cdot (-1)$$

$$\Rightarrow \boxed{i^4 = 1} \quad \underline{\hspace{2cm}} \quad (*)$$

$$\begin{array}{l}
 i^5 = i^4 \cdot i = i \\
 i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = -1 \\
 i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i \\
 i^8 = i^4 \cdot i^4 = 1 \cdot 1 = 1 \\
 i^9 = i^{4 \times 2} \cdot i = (i^4)^2 \cdot i = 1^2 \cdot i = i \\
 i^{10} = i^{4 \times 2} \cdot i^2 = (i^4)^2 \cdot (-1) = 1 \cdot (-1) = -1
 \end{array}$$

So we got

$$\left. \begin{array}{l}
 * \quad i^1 = i^5 = i^9 = i^{13} = \dots = i^{4n+1} \\
 i^2 = i^6 = i^{10} = i^{14} = \dots = i^{4n+2} \\
 i^3 = i^7 = i^{11} = i^{15} = \dots = i^{4n+3} \\
 i^4 = i^8 = i^{12} = i^{16} = \dots = i^{4n}
 \end{array} \right\} \underline{n \in \mathbb{N}}$$

* If there is a question find i^{50}

Method 1 $i^{50} = i^{4 \times 12 + 2} = i^2 = -1$

Method 2 or $i^{50} = i^{4 \times 12 + 2} = (i^4)^{12} \cdot i^2$
 $= 1^{12} \cdot (-1)$
 $= 1 \cdot (-1)$
 $= -1$

imaginary Numbers

OPERATIONS ON COMPLEX NUMBER (+, -, ×, ÷)

$$\text{Let } z_1 = a + ib$$

$$z_2 = c + id \text{ be two complex Numbers.}$$

Then

$$\begin{aligned} \text{(i) } z_1 + z_2 &= (a + ib) + (c + id) \\ &= (a + c) + (ib + id) \\ &= (a + c) + i(b + d) \end{aligned}$$

$$\begin{aligned} \text{(ii) } z_1 - z_2 &= (a + ib) - (c + id) \\ &= (a - c) + i(b - d) \end{aligned}$$

$$\begin{aligned} \text{(iii) } z_1 z_2 &= (a + ib)(c + id) \\ &= ac + iad + ibc + i^2 bd \\ &= ac + iad + ibc - bd \\ &= (ac - bd) + i(ad + bc) \end{aligned}$$

CONJUGATE AND MODULUS OF A COMPLEX NUMBER

Let $z = x + iy$ be a complex Number.

Then Modulus of z is denoted by $|z|$

and is defined as $|z| = |x + iy| = \sqrt{x^2 + y^2}$

$$\text{So } |z| = \sqrt{x^2 + y^2}$$

And conjugate of z is denoted by \bar{z}

and is defined as $\bar{z} = x - iy$

* That means conjugate is obtained by changing the sign of imaginary part.

for example

Q.1 find conjugate and Modulus of

$$z = 3 + 4i$$

Now given $z = 3 + 4i$, here $x = 3, y = 4$

$$\bar{z} = 3 - 4i$$

$$|z| = \sqrt{(3)^2 + (4)^2}$$

$$= \sqrt{9 + 16} = \sqrt{25} = 5$$

Q.4 Express in a+ib form i.e. $\frac{2+3i}{4-2i}$

Ans

$$\begin{aligned}Z &= \frac{2+3i}{4-2i} = \frac{(2+3i)(4+2i)}{(4-2i)(4+2i)} \\&= \frac{8 + 4i + 12i + 6i^2}{(4)^2 - (2i)^2} \\&= \frac{8 + 16i - 6}{16 - 4i^2} \quad (\because i^2 = -1) \\&= \frac{2 + 16i}{16 - 4(-1)} \\&= \frac{2 + 16i}{16 + 4} = \frac{2 + 16i}{20} \\&= \frac{2}{20} + \frac{16}{20}i \\&= \frac{1}{10} + \frac{4}{5}i\end{aligned}$$

Q.5 Find the conjugate of the complex no. $\frac{1}{3+4i}$

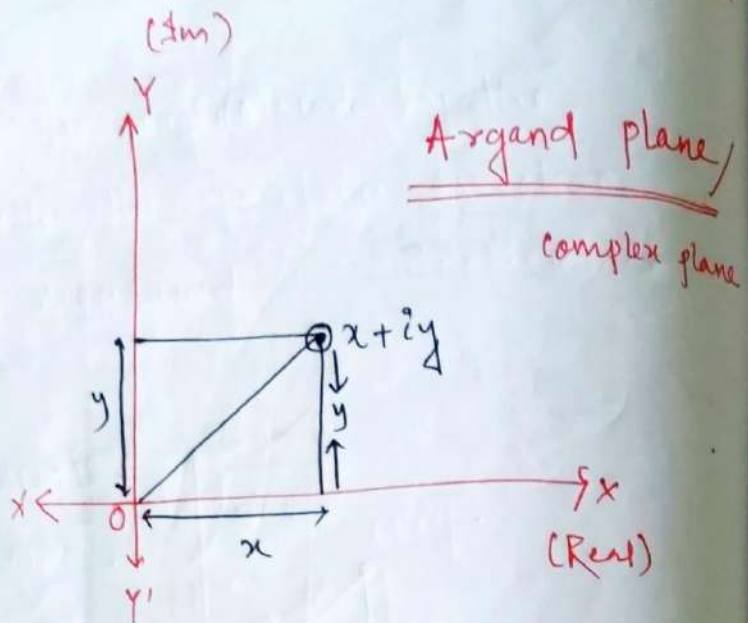
Solⁿ

$$\begin{aligned}Z &= \frac{1}{3+4i} \\&= \frac{1(3-4i)}{(3+4i)(3-4i)} = \frac{3-4i}{(3)^2 - (4i)^2} = \frac{3-4i}{9-16(-1)} \\&= \frac{3-4i}{9+16} = \frac{3-4i}{25} = \frac{3}{25} - \frac{4}{25}i\end{aligned}$$

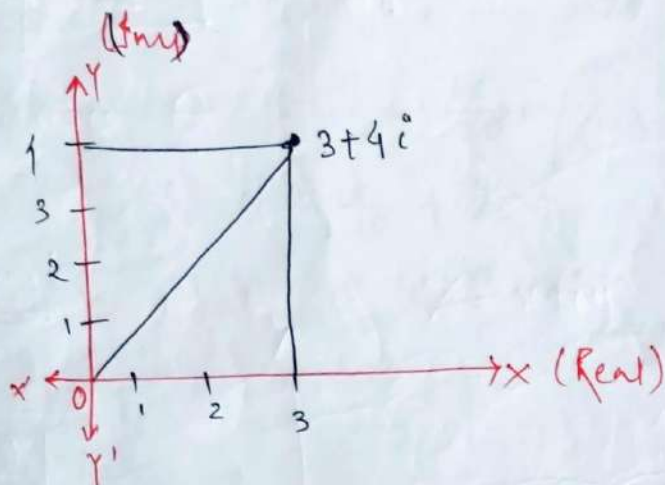
Then $\bar{Z} = \frac{3}{25} + \frac{4}{25}i$

Geometrical Representation of a complex Number

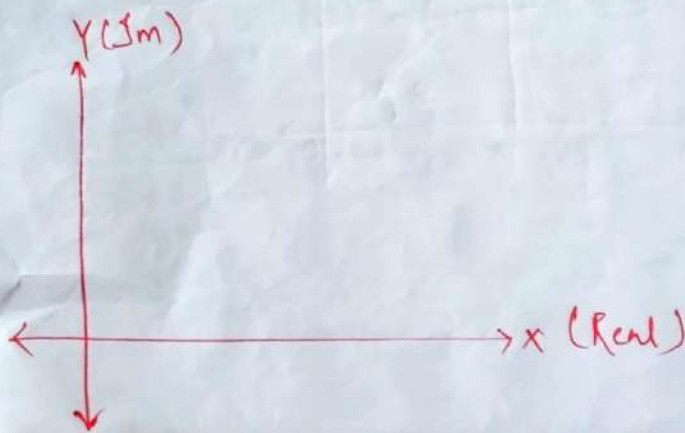
Let $z = x + iy$



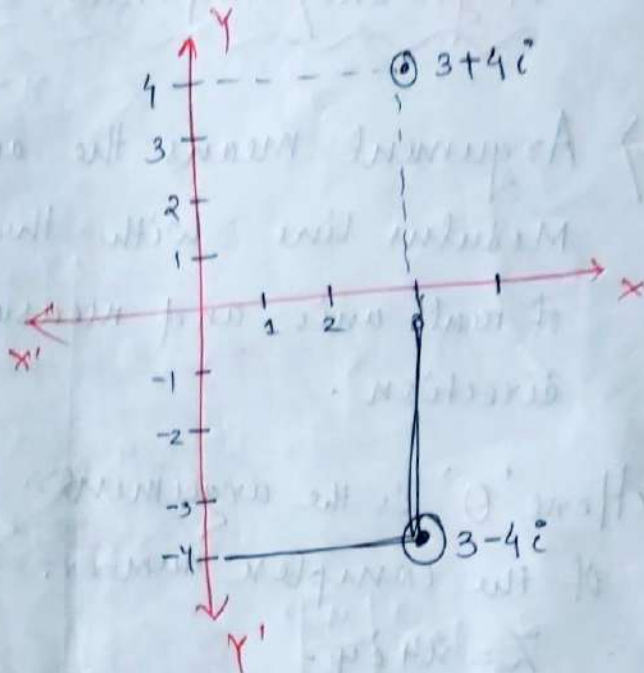
For example $z = 3 + 4i$



Example - 2 $z = 3 - 4i$



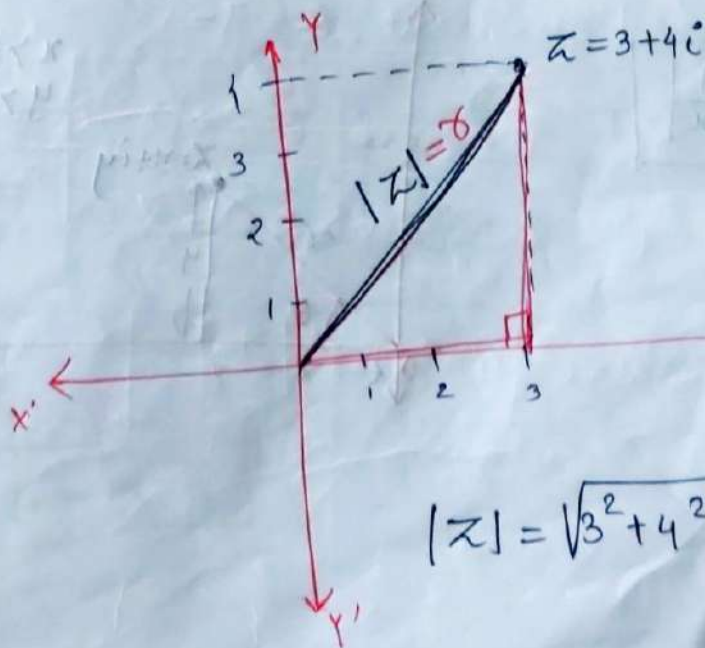
Example-2 $z = 3 - 4i$



NOTE conjugate of a complex Number is mirror image of that no. across the real axis.

⇒ Modulus of $z = 3 + 4i$

NOTE Modulus means the distance of the given complex number from the origin.



using pythagoras theorem.

$$h^2 = p^2 + b^2$$

$$|z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Argument / Amplitude of a complex Number.

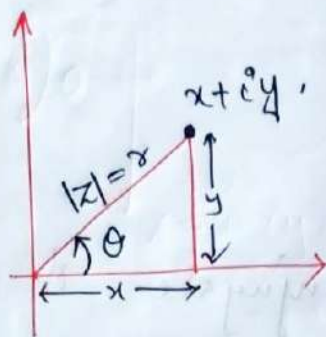
⇒ Argument means the angle (θ) made by the Modulus line with the positive direction of real axis and measured in anticlockwise direction.

⇒ Here ' θ ' is the argument of the complex number.

$$z = x + iy.$$

NOTE

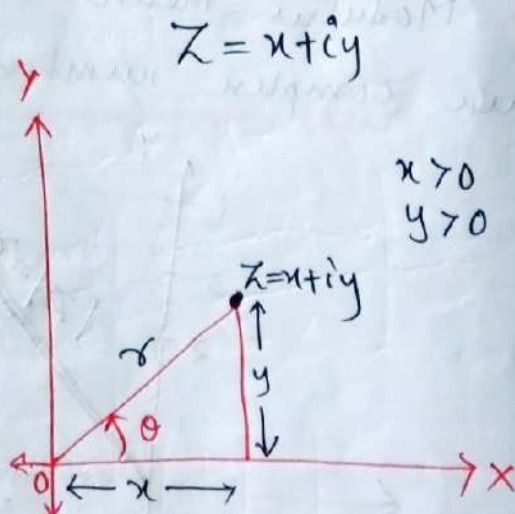
$$-\pi < \theta \leq \pi$$



How to measure θ

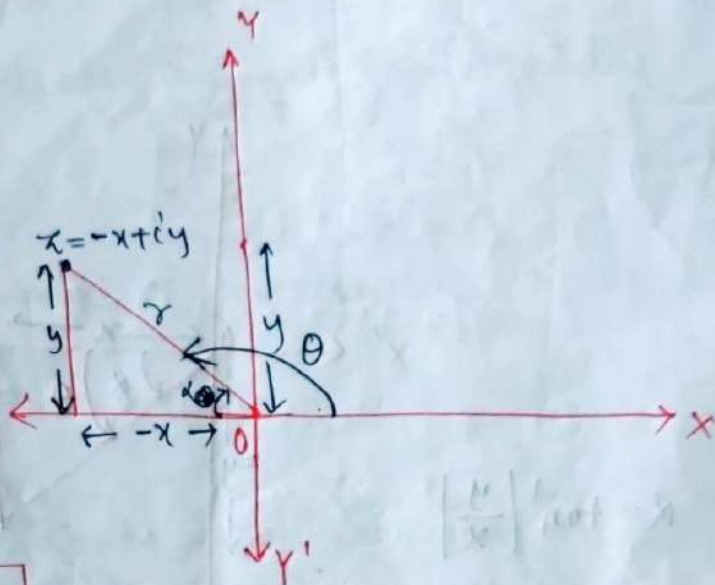
For 1st quadrant

$$\theta = \tan^{-1} \left| \frac{y}{x} \right|$$



For 2nd quadrant ($x < 0, y > 0$)

$$z = -x + iy$$

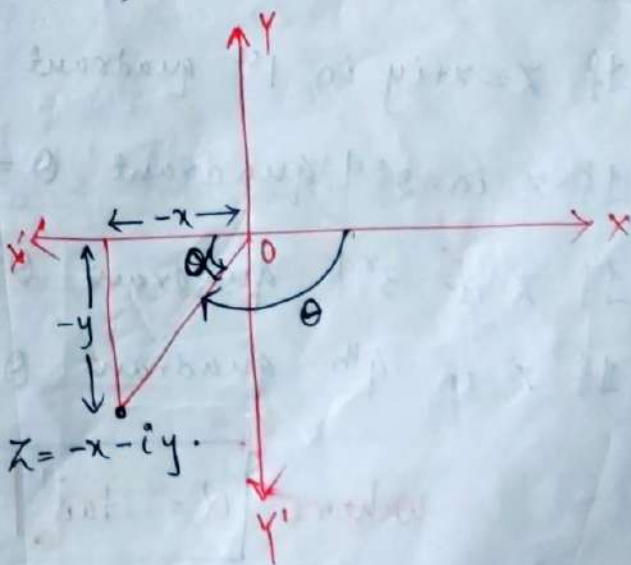


$$\alpha = \tan^{-1} \left| \frac{y}{x} \right|$$

$$\text{Now } \theta = \pi - \alpha$$

For 3rd quadrant

$z = -x - iy$
($x < 0, y < 0$)



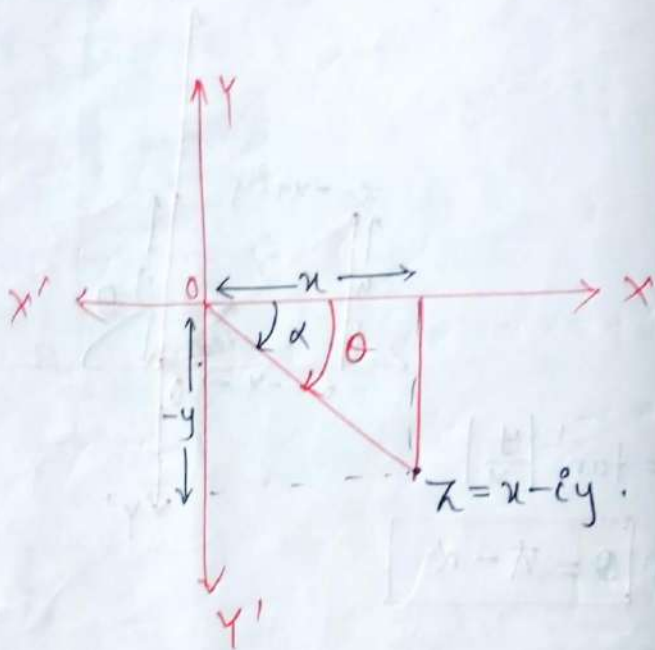
$$\alpha = \tan^{-1} \left| \frac{y}{x} \right|$$

$$\text{Then } \theta = -(\pi - \alpha)$$

$$\theta = \alpha - \pi$$

4th quadrant $x > 0, y < 0$

$$z = x - iy$$



$$\alpha = \tan^{-1} \left| \frac{y}{x} \right|$$

$$\theta = -\alpha$$

NOTE

- ① If $z = x + iy$ in 1st quadrant $\theta = \alpha$
- ② If z in 2nd quadrant $\theta = \pi - \alpha$
- ③ If z in 3rd quadrant $\theta = -(\pi - \alpha)$
- ④ If z in 4th quadrant $\theta = -\alpha$

where $\alpha = \tan^{-1} \left| \frac{y}{x} \right|$

CUBE ROOTS OF UNITY

Let cube root of unity is x .

$$\text{i.e. } \sqrt[3]{1} = x$$

$$\Rightarrow 1 = x^3$$

$$\Rightarrow x^3 - 1 = 0$$

$$\Rightarrow (x-1)(x^2+x+1) = 0$$

$$\Rightarrow \text{either } x-1=0$$

$$\text{or } x^2+x+1=0$$

$$\Rightarrow x=1$$

$$\text{or } x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{-3}}{2}$$

$$\Rightarrow x = \frac{-1 \pm i\sqrt{3}}{2}$$

So the cube roots of unity are . * one is real.

$$1, \quad \frac{-1 + i\sqrt{3}}{2} \quad \text{and} \quad \frac{-1 - i\sqrt{3}}{2}$$

other two roots are complex.

NOTE-1

Each complex cube root of unity is the square of the other.

$$\text{i.e. } \left(\frac{-1 + i\sqrt{3}}{2} \right)^2$$

$$= \frac{(-1)^2 + (i\sqrt{3})^2 + 2(-1)(i\sqrt{3})}{4}$$

$$= \frac{1 + i^2(\sqrt{3})^2 - 2i\sqrt{3}}{4}$$

$$= \frac{1 - 3 - 2i\sqrt{3}}{4} = \frac{-2 - 2i\sqrt{3}}{4} = \frac{2(-1 - i\sqrt{3})}{4}$$

$$= \frac{-1 - i\sqrt{3}}{2}$$

* So if we take

$$\omega = \frac{-1 + i\sqrt{3}}{2}$$

$$\text{then } \frac{-1 - i\sqrt{3}}{2} = \omega^2$$

NOTE-2

Sum of the cube roots of unity is zero.

$$\text{i.e. } \boxed{1 + \omega + \omega^2 = 0}$$

$$\text{or } \begin{cases} 1 + \omega = -\omega^2 \\ 1 + \omega^2 = -\omega \\ \omega + \omega^2 = -1 \end{cases}$$

$$\text{L.H.S } 1 + \frac{-1 + i\sqrt{3}}{2} + \frac{-1 - i\sqrt{3}}{2}$$

$$= \frac{2 - 1 + i\sqrt{3} - 1 - i\sqrt{3}}{2}$$

$$= \frac{2 - 2}{2} = \frac{0}{2} = 0 \text{ (R.H.S)}$$

NOTE-3

As 'w' is ^{one of} the cube root of unity

$$\text{i.e. } \sqrt[3]{1} = w$$

$$\Rightarrow \boxed{1 = w^3}$$

\Rightarrow So we got $w^3 = 1$

$$\text{Then } w^4 = w^3 \cdot w \\ = w$$

$$w^5 = w^3 \cdot w^2 \\ = w^2$$

$$w^6 = (w^3)^2 \\ = (1)^2 = 1$$

\Rightarrow $w^{56} = ?$

$$\begin{aligned} \text{So } w^{56} &= w^{3 \times 18 + 2} \\ &= w^{3 \times 18} \cdot w^2 \\ &= (w^3)^{18} \cdot w^2 \\ &= 1 \cdot w^2 = w^2 \end{aligned}$$

$$\begin{array}{r|l} 3 & 56 & | & 18 \\ & 3 & & \\ \hline & 26 & & \\ & 24 & & \\ \hline & 2 & & \end{array}$$

$$56 = 3 \times 18 + 2$$

De-Moivre's Theorem

Theorem - If n is an integer, positive or negative or zero, then

$$(\cos \theta + i \sin \theta)^n = (\cos n\theta + i \sin n\theta).$$

NOTE-1

$$\begin{aligned}(\cos \theta + i \sin \theta)^{-n} &= \cos(-n)\theta + i \sin(-n)\theta \\ &= \cos(-n\theta) + i \sin(-n\theta) \\ &= \cos n\theta - i \sin n\theta\end{aligned}$$

NOTE-2

$$\begin{aligned}(\cos \theta + i \sin \theta)^0 &= \cos(0 \cdot \theta) + i \sin(0 \cdot \theta) \\ &= \cos 0 + i \sin 0 \\ &= 1 + i \cdot 0 = 1\end{aligned}$$

Q:- find the value of $(\cos 60^\circ + i \sin 60^\circ)^3$

Ans:-

$$\begin{aligned}(\cos 60^\circ + i \sin 60^\circ)^3 &= \cos(3 \times 60^\circ) + i \sin(3 \times 60^\circ) \\ &= \cos 180^\circ + i \sin 180^\circ \\ &= -1 + i \cdot 0 \\ &= -1\end{aligned}$$

Square root of a complex Number :-

Q.1 Find the square roots of $3+4i$

Solⁿ Let $\sqrt{3+4i} = x+iy$

$$\Rightarrow 3+4i = (x+iy)^2$$

$$\Rightarrow 3+4i = x^2 + 2(x)(iy) + (iy)^2$$

$$\Rightarrow 3+4i = x^2 + 2ixy + i^2 y^2$$

$$\Rightarrow 3+4i = x^2 + 2ixy - y^2 \quad (\because i^2 = -1)$$

$$\Rightarrow 3+4i = x^2 - y^2 + i 2xy$$

Equating real and imaginary parts.

$$x^2 - y^2 = 3$$

$$\text{and } 2xy = 4$$

Now consider, $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$

$$\begin{aligned} \Rightarrow (x^2 + y^2)^2 &= (3)^2 + (2xy)^2 \\ &= 9 + (4)^2 = 9 + 16 \\ &= 25 \end{aligned}$$

~~$(a+b)^2 = a^2 + b^2 + 2ab$~~
 $\{ \because (a+b)^2 = (a-b)^2 + 4ab \}$

$$\Rightarrow x^2 + y^2 = \sqrt{25} = \pm 5 = 5$$

Solving

$$\text{Now } \wedge \quad x^2 + y^2 = 5$$

$$x^2 - y^2 = 3$$

$$2x^2 = 8$$

$$x^2 = 4$$

$$\Rightarrow x = \pm 2$$

putting value of x in $x^2 + y^2 = 5$

$$\Rightarrow (\pm 2)^2 + y^2 = 5$$

$$\Rightarrow 4 + y^2 = 5$$

$$\Rightarrow y^2 = 1$$

$$\Rightarrow y = \sqrt{1} = \pm 1$$

Hence Square roots of $3 + 4i = \pm 2 \pm 1i$
 $= \pm (2 + i)$

CHAPTER-2

MATRIX

Definition:-

Matrix is a rectangular array of numbers arranged in rows and columns.

e.g. $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$

order

→ If there are 'm' no. of rows and 'n' no. of columns, then the matrix is of order $m \times n$

e.g. $A = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 0 & 5 \end{bmatrix}_{2 \times 3}$

order = 2×3

2 = no. of rows

3 = no. of columns.

→ Sub Matrix

Any Matrix obtained by omitting some rows or columns or both of a given $m \times n$ Matrix is called its sub matrix.

e.g. $A = \begin{bmatrix} 2 & 0 & 4 \\ 5 & 2 & 1 \\ 7 & -1 & 4 \end{bmatrix}$

Then $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ is one of its

Sub-matrix

Determinant of a Matrix

⇒ Let A is a square Matrix of order 2

$$\text{i.e. } A = \begin{bmatrix} 4 & 5 \\ 1 & -6 \end{bmatrix}$$

Then Determinant of A is obtained by

$$\begin{aligned} |A| &= \begin{vmatrix} 4 & 5 \\ 1 & -6 \end{vmatrix} = -24 - 5 \\ &= -29 \end{aligned}$$

⇒ Let A is a square Matrix of order 3.

$$\text{i.e. } A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} - 2 \begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} + 3 \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}$$

$$= 1(28 - 30) - 2(21 - 25) + 3(18 - 20)$$

$$= 1(-2) - 2(-4) + 3(-2)$$

$$= -2 + 8 - 6 = 0$$

Rank of a Matrix

The rank is how many rows ^(or columns) of the matrix are unique (linearly independent).

e.g rank of $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \end{bmatrix}$

* Rank of a matrix A is denoted by $r(A)$

Solⁿ let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \end{bmatrix} \begin{matrix} \rightarrow R_1 \\ \rightarrow R_2 \end{matrix}$

Here $R_2 \rightarrow 3R_1$

So, here only R_1 is unique (linearly independent)

So $r(A) = 1$

Example 2 $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 4 & 5 \end{bmatrix}$

Here R_1 is unique

R_2 is unique (not dependent on R_1)

but $R_3 \rightarrow R_1 + R_2$ (i.e. dependent)

So $r(A) = 2$

Example 3 $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & -2 & -1 \end{bmatrix}$

Here R_1 is unique.

R_2 is also linearly independent.

$R_3 \rightarrow R_1 - 2R_2$ (linearly dependent)

$\Rightarrow r(A) = 2$

Another Method Using determinant. rank

A number ' r ' is said to be rank of a non-zero matrix if

* There exist at least one square submatrix ^{of order ' r '} whose determinant is not equal to zero.

* And determinant of every square matrix of order $(r+1)$ is zero.

Q:-1 Find the rank of $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Solⁿ Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

* As A is a Matrix of order 3×3

$$r(A) \leq 3$$

$$|A| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= 0 \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}$$

$$= +1 (0 - 2)$$

$$= -2 \neq 0$$

So rank of A or $r(A) = 3$

Q.2 Find the rank of the Matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Solⁿ Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$\rho(A) \leq 3$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(2-1) - 2(3-1) + 3(3-2)$$

$$= 1 - 2(2) + 3(1)$$

$$= 1 - 4 + 3$$

$$= 0$$

$$\text{So } \rho(A) \neq 3$$

Then consider any 2nd order sub-matrix,

$$\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = 2 - 6 = -4 \neq 0$$

$$\text{So } \rho(A) = 2$$

Another Method

Elementary Transformation

In elementary Transformation a Matrix is obtained by using (applying) a series of elementary row operations s.t. there are some non-zero rows on the top and the remaining rows consist of all zeros is called canonical Matrix.

⇒ Then the rank is the number of non-zero rows.

Elementary Operations

(1) Interchange of any two rows or columns.

for example

$$\begin{bmatrix} 3 & 0 & 2 \\ 4 & 2 & 1 \\ 5 & -2 & 5 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ 3 & 0 & 2 \\ 5 & -2 & 5 \end{bmatrix}$$

(2) Multiplication of any row by a non-zero number 'k'.

$$\begin{bmatrix} 4 & 2 \\ -5 & 4 \end{bmatrix}$$

$$R_2 \rightarrow 3R_2$$

$$= \begin{bmatrix} 4 & 2 \\ -15 & 12 \end{bmatrix}$$

(3) ~~Multiplication~~ Adding a row by Multiplying a non-zero number 'k' to another row.

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 0 & 2 \\ 5 & -1 & 4 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + 2R_3$$

$$= \begin{bmatrix} 2 & 3 & 1 \\ 14 & -2 & 10 \\ 5 & -1 & 4 \end{bmatrix}$$

$$\begin{array}{r} 4 \quad 0 \quad 2 \\ + \quad 10 \quad -2 \quad 8 \\ \hline 14 \quad -2 \quad 10 \end{array}$$

Find the rank of the followings.

(i)
$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$

Solⁿ
 let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}$

$R_2 \rightarrow R_2 - 3R_1, R_3 \rightarrow R_3 - 4R_1$

$\approx \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & -2 & -4 \end{bmatrix}$

$R_3 \rightarrow R_3 - R_2$

$\approx \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$

no. of non zero rows = 2

$\text{rank}(A) = 2$

v.v. Imp. NOTE (*)

Make below elements of diagonal as zero

i.e. Numbers that are rounded.

→ After that make other elements zero if possible without hamper

$R_2 = 3 \ 4 \ 5$ ing other zero

$3R_1 = 3(1 \ 2 \ 3)$ number

$= 3 \ 6 \ 9$

$R_2 - 3R_1$

$3 \ 4 \ 5$

$- 3 \ 6 \ 9$

(Ans) $0 \ -2 \ -4$

⇒ Rank of a Matrix is used to find solution of linear system of eq's.

Solution of linear system of eq's :- (Non-homogeneous)

(i) So, consider the set of 'm' ~~linear eq's~~ non-homogeneous linear equations.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = k_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = k_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = k_m$$

(i)

The above system of eq's can be written in ~~can~~ Matrix form i.e.

$$AX = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix}$$

here A is called coefficient Matrix.

$$\text{and } K = \left[\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & k_1 \\ a_{21} & \dots & a_{2n} & k_2 \\ a_{31} & \dots & a_{3n} & k_3 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & k_m \end{array} \right]$$

is called augmented Matrix.

NOTE 1:- The system of ^{linear} eq^s is said to be consistent if and only if the coefficient Matrix ' A ' and the augmented Matrix ' K ' are of same rank otherwise the system is inconsistent.

And the above statement is known as

Rouché's Theorem.

NOTE 2:- Let rank of $A = \pi$ And $n =$ no. of unknowns
rank of $K = \pi'$

Linear system of eq^s

(consistent or) solution ($\pi = \pi'$)

No solution ($\pi \neq \pi'$)
or inconsistent.

$\pi = \pi' = n$
(unique solⁿ)

$(\pi = \pi' < n)$
(infinitely many solⁿ)

(ii) Solution of linear system of eq's (Homogeneous)

Consider the 'n' number of homogeneous linear equations.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

$$\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

The above ~~Matrix~~ equations can be written in Matrix form.

$$AX = B$$

Where $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

As 'B' is a Zero Matrix.

So your Augmented Matrix (K) is equal to A

NOTE :-

(1) If rank of $A = n$ (no. of unknowns) after elementary row operations.

then the given linear eqⁿs have only a trivial solution.

$$\text{i.e. } x_1 = x_2 = \dots = x_n = 0$$

(2) If rank of $A < n$

then eq^s have infinitely many solutions

Q.1 solve $x - y + z = 0$ = xA

$$x + 2y - z = 0$$

$$2x + y + 3z = 0$$

Solution The given linear eq^s can be written

$$\text{in } AX = 0$$

where $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

consider $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

$$R_2 \rightarrow R_2 - R_1 \quad \& \quad R_3 \rightarrow R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 3 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\rho(A) = 3 = \text{no. of unknowns.}$$

Hence the eq^s have only trivial solutions

$$\text{i.e. } x = y = z = 0$$

Linear Differential Equations :-

Definition :-

Differential Equation in which the dependent variable (y) and all its derivatives are of degree '1' and are not multiplied together, is called a linear differential equation.

(i) Linear differential equation of order 1 :-

Linear differential equation of 1st order is given by

$$\boxed{\frac{dy}{dx} + py = Q}$$

where p and Q either functions of 'x' or constants.

(ii) Linear differential equation of nth order :-

Linear differential equation of nth order is given by

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + p_n y = X$$

where $p_1, p_2, p_3, \dots, p_n$ and X either functions of 'x' or constants.

Linear Differential equation of n th order with constant coefficients :-

L.D.E of n th order with constant coefficients is given by

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + k_n y = X \quad \text{--- (I)}$$

Where the coefficients k_1, k_2, \dots, k_n are constants.

* Now by using Operator ' D ' eqⁿ (I) can be written as.

$$D^n y + k_1 D^{n-1} y + k_2 D^{n-2} y + \dots + k_n y = X \quad \text{--- (II)}$$

Where $\boxed{D = \frac{d}{dx}}$ $\boxed{D^2 = \frac{d^2}{dx^2}}$ $\Rightarrow \frac{d^2 y}{dx^2} = D^2 y$

Then $\frac{dy}{dx} = Dy$

and so on...

Now eqⁿ (II) $\Rightarrow (D^n + k_1 D^{n-1} + k_2 D^{n-2} + \dots + k_n) y = X$
 $\Rightarrow f(D) y = X \quad \text{--- (III)}$

Where $f(D)$ represents a polynomial in D .

Solution of L.D.E of nth order with constant coefficients:-

A complete solution of

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + k_n y = X$$

is given by $y = u + v$

where 'u' is called complementary function (C.F.)

& is obtained both for homogeneous L.D.E & non-homogeneous L.D.E.

And 'v' is called particular Integral (P.I.)

which is obtained for only non-homogeneous L.D.E.

NOTE:-

(1) If $D^n y + k_1 D^{n-1} y + \dots + k_n y = 0$, it is called a homogeneous L.D.E of nth order.

(2) If $D^n y + k_1 D^{n-1} y + \dots + k_n y = X$, it is called a non-homogeneous L.D.E of nth order.

(3) For homogeneous L.D.E i.e. $f(D)y = 0$, solⁿ is $y = \text{C.F. only.}$

(4) for non-homogeneous L.D.E. i.e. $f(D)y = X$
Solⁿ is $\Rightarrow y = C.F + P.I.$

Rules for finding the complementary function:-

Given L.D.E is

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + k_n y = 0 \quad \text{--- (I)}$$

(i.e. a homogeneous L.D.E).

$$\Rightarrow D^n y + k_1 D^{n-1} y + k_2 D^{n-2} y + \dots + k_n y = 0$$

(by using operator D)

$$\Rightarrow (D^n + k_1 D^{n-1} + k_2 D^{n-2} + \dots + k_n) y = 0$$

$$\Rightarrow f(D)y = 0 \quad \text{--- (II)}$$

* Then by equating the symbolic coefficients to zero

$$\text{i.e. } D^n + k_1 D^{n-1} + k_2 D^{n-2} + \dots + k_n = 0$$

We will get the A.E. of ~~$f(D)y = 0$~~ eqⁿ (II)

i.e. $f(D) = 0$ is the Auxiliary equation of order 'n'

So by solving $f(D)=0$ ——— (III)

we will get 'n' no. of roots.

i.e. let m_1, m_2, \dots, m_n .

Then there ~~are~~ arises four cases depending upon the roots.

Case I :- If the roots are real and unequal.

If the roots will different

Then eqⁿ (II) i.e. $f(D)y=0$ becomes.

$$\Rightarrow (D-m_1)(D-m_2) \dots (D-m_n)y=0$$

Now taking $(D-m_1)y=0$

$$\Rightarrow Dy - m_1y = 0$$

$\Rightarrow \frac{dy}{dx} - m_1y = 0$ which is of the form

$$\boxed{\frac{dy}{dx} + py = Q}$$

then its solⁿ is obtained as follows:

Solⁿ
Given $\frac{dy}{dx} - m_1 y = 0$

here $P = -m_1$

$Q = 0$

then I.f = $e^{\int P dx}$

$= e^{\int -m_1 dx}$

$= e^{-m_1 x}$

then solⁿ is given by

$y \times \text{I.f} = \int (Q \times \text{I.f}) dx$

$\Rightarrow y e^{-m_1 x} = \int 0 dx$

$\Rightarrow y e^{-m_1 x} = c_1$

$\Rightarrow y = c_1 e^{m_1 x}$

then for the factor $(D - m_2)y = 0$

$y = c_2 e^{m_2 x}$

for $(D - m_3)y = 0$

$\Rightarrow y = c_3 e^{m_3 x}$

and so -- on --

Then the complete solⁿ of $f(D)y=0$

$$\text{is } y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$$

NOTE :- If the roots are real & unequal.

Then

$$\text{C.F} = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$$

Where m_1, m_2, \dots, m_n are the roots.

Case II :- If the roots are real and equal.

Let m_1, m_2, \dots, m_n are the real roots.

(I) where $m_1 = m_2$ and m_3, m_4, \dots, m_n are different

$$\text{then } \text{C.F} = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

(II) If $m_1 = m_2 = m_3$ and m_4, m_5, \dots, m_n are different.

$$\text{then } \text{C.F} = (c_1 x^2 + c_2 x + c_3) e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$$

Case III :- If roots are complex and unequal.

$$\text{let } m_1 = \alpha + i\beta$$

$$m_2 = \alpha - i\beta \text{ are two complex roots}$$

and m_3, m_4, \dots, m_n are real & different roots.

Then

$$C.F = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$$

Case-IV :- If the roots are complex and equal.

$$\text{let } m_1 = m_2 = \alpha + i\beta$$

$$m_3 = m_4 = \alpha - i\beta$$

and m_5, m_6, \dots, m_n are real & different roots.

Then .

$$C.F = e^{\alpha x} [(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x] + c_5 e^{m_5 x} + \dots$$

Q.1 Solve $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0$

Solution Given L.D.E is

$$\Rightarrow \frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0 \quad \text{--- (1)}$$

using Operator D

$$\Rightarrow D^2y + 7Dy + 12y = 0$$

$$\Rightarrow (D^2 + 7D + 12)y = 0 \quad \text{--- (11)}$$

Auxiliary Equation of eqⁿ (11) is given by

$$\Rightarrow D^2 + 7D + 12 = 0$$

$$\text{Then } D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-7 \pm \sqrt{(7)^2 - 4 \cdot 1(12)}}{2(1)}$$

$$= \frac{-7 \pm 1}{2}$$

$$\text{So } D = \frac{-7+1}{2} \text{ or } \frac{-7-1}{2}$$

$$= -3 \text{ or } -4$$

So here the roots -3 & -4 are real and distinct.

So C.F = $C_1 e^{-3x} + C_2 e^{-4x}$ which is the complete solution of eqⁿ(1).

Q.2 solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

Solution Given $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$ — (i)

$\Rightarrow D^2y - 2Dy + y = 0$ (using operator D)

$\Rightarrow (D^2 - 2D + 1)y = 0$ — (ii)

Auxiliary eqⁿ is given by

$\Rightarrow D^2 - 2D + 1 = 0$ — (iii)

Solving eqⁿ (iii) $D = \frac{2 \pm \sqrt{4-4}}{2}$

$= \frac{2}{2}$

$D = 1$ or 1 (as $D^2 - 2D + 1$ is a quadratic eqⁿ, so we will get 2 roots)

Here the roots $\neq 1$ are real & equal

So C.F. = $(C_1x + C_2)e^x$ is the complete solⁿ of eqⁿ (1).

Q.3 Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

Solution Given L.D.E is $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$ — (1)

$\Rightarrow D^2y + 4Dy + 13y = 0$ (using Operator D)

$\Rightarrow (D^2 + 4D + 13)y = 0$ — (II)

A.E. is given by

$D^2 + 4D + 13 = 0$ — (III)

Now solving (III) i.e. $D = \frac{-4 \pm \sqrt{16 - 52}}{2}$

$\sqrt{36} = 6$
but $\sqrt{-36}$
 $= \sqrt{(-1)36}$
 $= \sqrt{i^2 36}$
 $= i6$ as $-1 = i^2$

$= \frac{-4 \pm \sqrt{-36}}{2}$

$= \frac{-4 \pm i6}{2}$

So $D = \frac{-4 + 6i}{2}$ or $\frac{-4 - 6i}{2}$

$$\Rightarrow D = -2 + 3i \quad \text{or} \quad -2 - 3i$$

$$\text{here } \alpha = -2$$

$$\beta = 3$$

$$\text{then C.F.} = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

$$= e^{-2x} (C_1 \cos 3x + C_2 \sin 3x)$$

Which is the complete solution.

Q.4 solve $\frac{d^3 y}{dx^3} + y = 0$

Solution :- Given $\frac{d^3 y}{dx^3} + y = 0$ — (I)

$$\Rightarrow D^3 y + y = 0 \quad (\text{using Operator } D)$$

$$\Rightarrow (D^3 + 1)y = 0 \quad \text{--- (II)}$$

Then A.E. is given by

$$D^3 + 1 = 0$$

$$\text{Solving } D^3 + 1 = 0$$

$$\Rightarrow (D)^3 + (1)^3 = 0$$

$$\Rightarrow (D+1)(D^2 - D + 1) = 0$$

$$\text{If } D+1=0 \Rightarrow \boxed{D=-1}$$

$$\text{if } D^2-D+1=0$$

$$\Rightarrow D = \frac{1 \pm \sqrt{1-4(1)(1)}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

$$= \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\text{So } \boxed{D = \frac{1}{2} + i\frac{\sqrt{3}}{2}} \quad \& \quad \boxed{D = \frac{1}{2} - i\frac{\sqrt{3}}{2}}$$

Here -1 , $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ & $\frac{1}{2} - i\frac{\sqrt{3}}{2}$ are the roots
 \downarrow real root. \hookrightarrow complex roots \leftarrow

$$\text{then C.F} = C_1 e^{-x} + e^{\frac{1}{2}x} \left(C_2 \cos \frac{\sqrt{3}}{2} + C_3 \sin \frac{\sqrt{3}}{2} \right)$$

is the complete solution.

Rules to find Particular Integral (P.I.)

To find P.I. of L.D.E. we need an Inverse differential operator.

Inverse differential Operator :-

We already have discussed differential operators

$$\text{that is } D = \frac{d}{dx}.$$

And the Inverse Operator is denoted by $\frac{1}{D}$

which stands for Integral.

Now consider the eqⁿ.

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + k_n y = X \quad \text{--- (1)}$$

$$\Rightarrow D^n y + k_1 D^{n-1} y + k_2 D^{n-2} y + \dots + k_n y = X$$

(using differential operator)

$$\Rightarrow (D^n + k_1 D^{n-1} + k_2 D^{n-2} + \dots + k_n) y = X$$

$$\Rightarrow f(D) y = X$$

$$\Rightarrow y = \frac{1}{f(D)} X, \text{ which is the P.I. of eqⁿ (1).}$$

Then there arises four cases depending upon the 'X'

Case I when $X = e^{ax}$ (i.e. an exponential function)

$$\text{Then P.I} = \frac{1}{f(D)} e^{ax}$$

$$= \frac{1}{f(a)} e^{ax}, \text{ provided } f(a) \neq 0.$$

If $f(a) = 0$

$$\text{Then P.I} = \frac{1}{f(D)} e^{ax}$$

$$= \frac{x}{f'(a)} e^{ax}, \text{ provided } f'(a) \neq 0.$$

$$\text{If } f'(a) = 0 \text{ then P.I} = \frac{x^2}{f''(a)} e^{ax}, \text{ provided } f''(a) \neq 0$$

and so on.

Q.1 find P.I. of $(D^2 + 5D + 6)y = e^x$

Solution :- Given $(D^2 + 5D + 6)y = e^x$

$$\Rightarrow P.I = \frac{1}{D^2 + 5D + 6} e^x.$$

put $D=1$

$$\Rightarrow P.I = \frac{1}{12} e^x.$$

(Ans).

Q.2 find P.I. of $(D-2)^2 y = e^{2x}$.

Solution :- Given $(D-2)^2 y = e^{2x}$

$$\Rightarrow P.I = \frac{1}{(D-2)^2} e^{2x}$$

$$= \frac{1}{D^2 - 4D + 4} e^{2x}$$

Putting $D=2$ $f(D)=0$, so case failure
~~so find $f'(D)$.~~

$$= \frac{x}{2D-4} e^{2x}$$

Putting $D=2$ $f'(D)=0$, so case failure

$$= \frac{x^2}{2} e^{2x} \quad (\text{Ans}).$$

Case II :- If $X = \sin(ax+b)$ or $\cos(ax+b)$.

$$\text{Then P.I} = \frac{1}{f(D^2)} \sin(ax+b)$$

$$= \frac{1}{f(-a^2)} \sin(ax+b), \text{ provided } f(-a^2) \neq 0.$$

If $f(-a^2) = 0$, then rule fails.

$$\text{P.I} = \frac{x}{f'(D^2)} \sin(ax+b).$$

$$= \frac{x}{f'(-a^2)} \sin(ax+b), \text{ provided } f'(-a^2) \neq 0$$

and so ... on.

Q:- find P.I. of $(D^2 + D + 1)y = \sin 2x$.

Solution :-

$$\text{P.I} = \frac{1}{D^2 + D + 1} \sin 2x.$$

put $-a^2$ in place of D^2

$$= \frac{1}{-4 + D + 1} \sin 2x.$$

$$= \frac{1}{D-3} \sin 2x.$$

$$= \frac{(D+3)}{(D-3)(D+3)} \sin 2x$$

$$= \frac{D+3}{D^2-9} \sin 2x$$

put $-a^2$ in place of D^2

$$= \frac{D+3}{-4-9} \sin 2x$$

$$= \frac{D+3}{-13} \sin 2x$$

$$= -\frac{1}{13} [D \sin 2x + 3 \sin 2x]$$

$$= -\frac{1}{13} [2 \cos 2x + 3 \sin 2x] \quad (\text{Ans})$$

Q.2 find the P.I. of $\frac{d^3y}{dx^3} + 4 \frac{dy}{dx} = \sin 2x$.

Solⁿ Given eqⁿ is $D^3y + 4Dy = \sin 2x$

$$\Rightarrow (D^3 + 4D)y = \sin 2x$$

$$\text{P.I} = \frac{1}{D^3 + 4D} \sin 2x$$

$$= \frac{1}{D^2 \cdot D + 4D} \sin 2x.$$

Put $-a^2$ in place of D^2

$$= \frac{1}{-4D + 4D} \sin 2x \quad \text{case failure,}$$

$$= \frac{x}{3D^2 + 4} \sin 2x.$$

Put $-a^2$ in place of D^2

$$= \frac{x}{3(-4) + 4} \sin 2x.$$

$$= \frac{x}{-8} \sin 2x \quad (\text{Ans}).$$

Case III:- when $X = x^m$

$$P.I = \frac{1}{f(D)} x^m.$$

$$= [f'(D)]^{-1} x^m.$$

Here we are going to use two binomial expansion.

$$(i) (1+D)^{-1} = 1 - D + D^2 - D^3 + \dots$$

$$(ii) (1-D)^{-1} = 1 + D + D^2 + D^3 + \dots$$

Q.2 Find P.I. of $(D^2+3D+2)y=x^2$

Solⁿ P.I. = $\frac{1}{D^2+3D+2} x^2$

$$= [D^2+3D+2]^{-1} x^2$$

$$= \left[2 \left(1 + \frac{D^2+3D}{2} \right) \right]^{-1} x^2$$

$$= \frac{1}{2} \left[1 + \frac{D^2+3D}{2} \right]^{-1} x^2$$

$$= \frac{1}{2} \left[1 - \frac{D^2+3D}{2} + \left(\frac{D^2+3D}{2} \right)^2 - \dots \right] x^2$$

$$= \frac{1}{2} \left[x^2 - \frac{D^2+3D}{2} (x^2) + \frac{(D^2+3D)^2}{2} x^2 \right]$$

$$= \frac{1}{2} \left[x^2 - \left(\frac{2+6x}{2} \right) + \left(\frac{0+18+0}{2} \right) \right]$$

$$= \frac{1}{2} \left[x^2 - (1+3x) + 9 \right]$$

$$= \frac{1}{2} \left[x^2 - 3x + 8 \right] \text{ (Ans.)}$$

Case-IV :- When $X = e^{ax} V$, where
 V being a function of x .

$$\begin{aligned}\text{Then P.I} &= \frac{1}{f(D)} e^{ax} V \\ &= e^{ax} \frac{1}{f(D+a)} V.\end{aligned}$$

Q.1 find P-I. of $(D^2 - 2D + 4) y = e^x \cos x$.

Solution :- P-I = $\frac{1}{D^2 - 2D + 4} e^x \cos x$.

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x.$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 4} \cos x,$$

$$= e^x \frac{1}{D^2 + 3} \cos x.$$

~~$= e^x$~~ put $D^2 = -a^2 = -1$

$$= e^x \frac{1}{-1+3} \cos x,$$

$$= \frac{e^x}{2} \cos x \text{ (Ans)}$$

Q.2 Find P-I. of $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x} \sin 2x$.

Solⁿ Given $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x} \sin 2x$.

$$\Rightarrow D^2y + 5Dy + 6y = e^{-2x} \sin 2x,$$

$$\Rightarrow (D^2 + 5D + 6)y = e^{-2x} \sin 2x,$$

$$\text{P-I} = \frac{1}{D^2 + 5D + 6} e^{-2x} \sin 2x$$

$$= e^{-2x} \frac{1}{(D-2)^2 + 5(D-2) + 6} \sin 2x.$$

$$= e^{-2x} \frac{1}{D^2 - 4D + 4 + 5D - 10 + 6} \sin 2x.$$

$$= e^{-2x} \frac{1}{D^2 + D} \sin 2x.$$

$$\text{put } D^2 = -a^2 = -4$$

$$= e^{-2x} \frac{1}{-4 + D} \sin 2x.$$

$$= e^{-2x} \frac{D+4}{(D-4)(D+4)} \sin 2x.$$

$$= e^{-2x} \frac{D+4}{D^2 - 16} \sin 2x.$$

$$= \frac{e^{-2x} (D+4)}{-4-16} \sin 2x,$$

$$= \frac{e^{-2x}}{-20} \sin 2x = \frac{e^{-2x}}{-20} (D \sin 2x + 4 \sin 2x)$$

$$\text{(Ans.) } = \frac{e^{-2x}}{-20} (2 \cos 2x + 4 \sin 2x)$$

(Ans.)

Q.1 Solve

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + \sin 2x$$

Solⁿ Given diff eqⁿ is

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + \sin 2x.$$

$$\Rightarrow D^3y + 2D^2y + Dy = e^{2x} + \sin 2x \quad (\text{using diff-operator } D)$$

$$\Rightarrow (D^3 + 2D^2 + D)y = e^{2x} + \sin 2x.$$

Then A.E is given by.

$$D^3 + 2D^2 + D = 0$$

$$\Rightarrow D(D^2 + 2D + 1) = 0$$

So either $D=0$ or $D^2 + 2D + 1 = 0$

$$\Rightarrow D = -1, -1$$

So the roots are $D=0, -1, -1$.

$$\text{Then C.F} = C_1 e^{0 \cdot x} + (C_2 x + C_3) e^{-1(x)}$$

$$= C_1 + (C_2 x + C_3) e^{-x}.$$

$$\text{And P.I} = \frac{1}{D^3 + 2D^2 + D} (e^{2x} + \sin 2x)$$

$$= \frac{1}{D^3 + 2D^2 + D} e^{2x} + \frac{1}{D^3 + 2D^2 + D} \sin 2x$$

P-I₁

P-I₂

$$\text{P.I}_1 = \frac{1}{D^3 + 2D^2 + D} e^{2x}$$

$$= \frac{1}{(2)^3 + 2(2)^2 + 2} e^{2x}$$

$$= \frac{e^{2x}}{18}$$

$$\text{P.I}_2 = \frac{1}{D^3 + 2D^2 + D} \sin 2x$$

$$= \frac{1}{D^2 - D + 2D^2 + D} \sin 2x$$

$$\text{Put } D^2 = -a^2 = -4$$

$$= \frac{1}{-4D + (-8) + D} \sin 2x$$

$$= \frac{1}{-3D-8} \sin 2x.$$

$$= \frac{-3D+8}{(-3D-8)(-3D+8)} \sin 2x$$

$$= \frac{-3D+8}{(-3D)^2 - (8)^2} \sin 2x,$$

$$= \frac{-3D+8}{9D^2 - 64} \sin 2x.$$

put $D^2 = -4$

$$= \frac{-3D+8}{-36-64} \sin 2x,$$

$$= \frac{-3D+8}{-100} \sin 2x.$$

$$= \frac{-4}{100} (-3D \sin 2x + 8 \sin 2x)$$

$$= \frac{-1}{100} (-3(2 \cos 2x) + 8 \sin 2x).$$

Linear Partial Differential Equation of first order (Lagrange's form)

Lagrange's linear equation is of the form

$$\boxed{Pp + Qq = R} \quad \text{--- (1) (standard form)}$$

where P, Q & R are functions of x, y, z and this is called a quasi-linear eqⁿ.

$$\text{and } p = \frac{\partial z}{\partial x} \quad \& \quad q = \frac{\partial z}{\partial y}$$

Solution of linear P.D.E of first order.

⇒ first find A.E for eqⁿ (1)

$$\text{i.e. } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{--- (2)}$$

⇒ and we have to solve the above A.E.

and this can be done by using two Methods.

(i) Grouping Method.

(ii) Method of Multipliers,

(iii) combination of (i) & (ii)

→ Suppose $U=a$ and $V=b$ are two sol's of A.E by using any of the Methods where U and V are functions of x, y, z .

→ Then complete solution of eqⁿ (I) is

$$f(a, b) = 0$$

$$\text{or } f(u, v) = 0$$

$$\text{or } u = \phi(v) \text{ or } v = \phi(u).$$

① Grouping Method :-

we have to make groups from eqⁿ (II)

which can be easily integrable.

$$\text{like make } \frac{dx}{P} = \frac{dy}{Q} \text{ (one group)}$$

and by solving the differential eqⁿ.

we get $u = a$ (one solution)

$$\text{again we can make } \frac{dy}{Q} = \frac{dz}{R} \text{ (another group)}$$

and by solving above

we will get $u = b$ (2nd solution)

(ii) Method of Multipliers :-

In this Method choose Multipliers l, m, n

Such that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l dx + m dy + n dz}{Pl + Qm + Rn}$$
$$= \frac{l dx + m dy + n dz}{0}$$

Choose l, m, n s.t. the ~~fourth~~ denominator of fourth part of A.E will be zero.

\times $l dx + m dy + n dz$ will be integrate easily.

Let's make a group,

$$\Rightarrow \frac{dx}{P} = \frac{l dx + m dy + n dz}{0}$$

$$\Rightarrow l dx + m dy + n dz = 0 \text{ ————— ①}$$

And Integrating the eqⁿ ① we will get a solⁿ $u = a$

\rightarrow Similarly choose another set of multipliers

s.t. we will get another solⁿ $v = b$.

Then complete solⁿ is $f(u, v) = 0$

③ Combination of ① & ② :-

In some cases we will get a solⁿ from grouping method and another solⁿ is obtained from method of multipliers.

Q.1 solve $P\sqrt{x} + Q\sqrt{y} = \sqrt{z}$

Solⁿ

Given P.D.E $P\sqrt{x} + Q\sqrt{y} = \sqrt{z}$ — (1)
(which is a Partial L.D.E in Lagrange's form).

\Rightarrow Then A.E. of (1) is given by

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}} \quad \left(\begin{array}{l} \text{as } P = \sqrt{x} \\ Q = \sqrt{y} \\ R = \sqrt{z} \end{array} \right)$$

Considering first two ratios, and make a group.

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}}$$

$$\Rightarrow \int \frac{dx}{\sqrt{x}} = \int \frac{dy}{\sqrt{y}}$$

$$\Rightarrow \int x^{-1/2} dx = \int y^{-1/2} dy.$$

$$\Rightarrow 2\sqrt{x} = 2\sqrt{y} + C_1,$$

$$\Rightarrow \sqrt{x} - \sqrt{y} = C_1 \text{ — (1)}$$

Again considering last two ratios and make a group

$$\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

$$\Rightarrow \int \frac{dy}{\sqrt{y}} = \int \frac{dz}{\sqrt{z}}$$

$$\Rightarrow 2\sqrt{y} = 2\sqrt{z} + 2C_2$$

$$\Rightarrow \sqrt{y} - \sqrt{z} = C_2 \text{ — (2)}$$

So the required solⁿ.

$$\phi(\sqrt{x}-\sqrt{y}, \sqrt{y}-\sqrt{z})=0$$

$$\text{or } \sqrt{x}-\sqrt{y}=f(\sqrt{y}-\sqrt{z})$$

Q.2 Solve $\frac{y^2z}{x}P + xzQ = y^2$

Solⁿ $y^2zP + x^2zQ = y^2x$ — (1)

Given Partial L.D.E is in Lagrange's form.

Here $P = y^2z$, $Q = x^2z$ and $R = y^2x$.

Then A.E for (1) is given by $\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{y^2x}$

Take first two ratios and make a group.

$$\frac{dx}{y^2x} = \frac{dy}{x^2x}$$

$$\Rightarrow x^2dx = y^2dy$$

Integrate both sides.

$$\Rightarrow \int x^2dx = \int y^2dy$$

$$\Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + C_1$$

$$\Rightarrow \frac{x^3}{3} - \frac{y^3}{3} = C_1$$

$$\Rightarrow x^3 - y^3 = C_1 \text{ — (II)}$$

Again take first and third ratio to make another group.

$$\frac{dx}{y^2x} = \frac{dz}{y^2x}$$

$$\Rightarrow \frac{dx}{x} = \frac{dz}{z}$$

$$\Rightarrow xdx = zdz$$

Integrate both sides.

$$\Rightarrow \int xdx = \int zdz$$

$$\Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + C_2$$

$$\Rightarrow \frac{x^2}{2} - \frac{z^2}{2} = C_2$$

$$\Rightarrow x^2 - z^2 = 2C_2$$

$$\Rightarrow x^2 - z^2 = C_2 \text{ — (IV)}$$

So the complete solⁿ is $\phi(x^3 - y^3, x^2 - z^2) = 0$

Q.1 Solve $x(y-z)P + y(z-x)Q = z(x-y)$

Solution Here $P = x(y-z)$

$$Q = y(z-x)$$

$$R = z(x-y)$$

Then auxiliary Eqⁿ is given by

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \quad \text{--- (1)}$$

We can solve this A.E. by using Method of Multipliers.

Choose ~~the~~ multipliers $l=1, m=1$ & $n=1$ and make 4th ratio,

Then $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{1dx + 1dy + 1dz}{x(y-z) + y(z-x) + z(x-y)}$

Now make a group by taking 1st & 4th ratio

$$\Rightarrow \frac{dx}{x(y-z)} = \frac{dx + dy + dz}{x(y-z) + y(z-x) + z(x-y)}$$

$$\Rightarrow dx + dy + dz = 0 \quad (\because x(y-z) + y(z-x) + z(x-y) \\ \neq xy - xz + yz - yx + zx - yz \\ = 0)$$

Now Integrating both sides.

$$\Rightarrow \int dx + \int dy + \int dz = \int 0$$

$$\Rightarrow x + y + z = c_1 \quad \text{--- (1)}$$

Again choose another set of Multipliers.

i.e. $l' = \frac{1}{x}$, $m' = \frac{1}{y}$, $n' = \frac{1}{z}$ and make

4th ratio from eq (1)

$$\text{Then } \frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y-z) + (z-x) + (x-y)}$$

Now make a group by taking 3rd & 4th ratio

$$\Rightarrow \frac{dz}{z(x-y)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y-z) + (z-x) + (x-y)}$$

$$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0 \quad (\because y-z + z-x + x-y \\ = 0)$$

Now.. Integrating both sides.

$$\Rightarrow \int \frac{1}{x} dx + \int \frac{1}{y} dy + \int \frac{1}{z} dz = \cancel{C_1} C_2$$

$$\Rightarrow \log x + \log y + \log z = C_2$$

$$\Rightarrow \log(xyz) = C_2$$

$$\Rightarrow xyz = e^{C_2} = C_3 \quad \text{--- (11)}$$

So the complete solⁿ is

$$f(x+y+z, xyz) = 0$$

Q:-2 Solve $x(y^2 - z^2)p + y(z^2 - x^2)q - (x^2 - y^2)z = 0$

Given P.D.E is

$$x(y^2 - z^2)p + y(z^2 - x^2)q = (x^2 - y^2)z$$

where $P = x(y^2 - z^2)$

$$Q = y(z^2 - x^2)$$

$$R = z(x^2 - y^2)$$

Then A.E is

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} \quad \text{--- (1)}$$

Choose multipliers $l = \frac{1}{x}$, $m = \frac{1}{y}$ & $n = \frac{1}{z}$
and make another ratio from eqⁿ (1)

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} = \frac{\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}{(y^2 - z^2) + (z^2 - x^2) + (x^2 - y^2)}$$

Choose 1st & 4th ratio, make a group.

$$\Rightarrow \frac{dx}{x(y^2 - z^2)} = \frac{\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}{(y^2 - z^2) + (z^2 - x^2) + (x^2 - y^2)}$$

$$\Rightarrow \frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = 0 \quad (\because y^2 - z^2 + z^2 - x^2 + x^2 - y^2 = 0)$$

Integrating both sides,

$$\Rightarrow \int \frac{1}{x} dx + \int \frac{1}{y} dy + \int \frac{1}{z} dz = C_1$$

$$\Rightarrow \log x + \log y + \log z = C_1$$

$$\Rightarrow \log (xyz) = C_1$$

$$\Rightarrow xyz = C_2 \quad \text{--- (II)}$$

choose another set of multipliers.

let $l' = x$, $m' = y$, $n' = z$ then 4th ratio of

A.F. will be.

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} = \frac{x dx + y dy + z dz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)}$$

then taking 3rd & 4th ratio

$$\Rightarrow \frac{dz}{z(x^2 - y^2)} = \frac{x dx + y dy + z dz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)}$$

$$\Rightarrow x dx + y dy + z dz = 0 \quad \left(\because x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2) = 0 \right)$$

Integrating,

$$\Rightarrow \int x dx + \int y dy + \int z dz = C$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C$$

$$\Rightarrow x^2 + y^2 + z^2 = 2C = C_3 \quad \text{--- (iii)}$$

So the complete soln is

$$f(xyz, x^2 + y^2 + z^2) = 0$$

LAPLACE TRANSFORMS

Let $f(t)$ be a function of 't'.

Then the Laplace Transform of $f(t)$ is denoted as $L\{f(t)\}$, is defined as.

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

provided integration exists.

* $L\{f(t)\} = F(s)$, s is a parameter
may be real or complex.

Laplace Transforms of some elementary functions

$$(1) L\{1\} = \frac{1}{s}$$

$$(2) L\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = \text{integer.}$$

$$(3) L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}, \quad n \text{ is fraction.}$$

$$(4) L\{e^{at}\} = \frac{1}{s-a}$$

$$(5) L\{\sin at\} = \frac{a}{s^2+a^2}$$

$$(6) L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$(7) L\{\sinh at\} = \frac{a}{s^2-a^2}$$

$$(8) L\{\cosh at\} = \frac{s}{s^2-a^2}$$

Proof (i): -

$$\begin{aligned} \underline{\text{L.H.S}} \Rightarrow L\{1\} &= \int_0^{\infty} e^{-st} \cdot 1 \, dt \\ &= \left. \frac{e^{-st}}{-s} \right]_0^{\infty} = -\frac{1}{s} \left\{ \frac{1}{e^{st}} \right\}_0^{\infty} \end{aligned}$$

$$= -\frac{1}{s} (0 - 1)$$

$$= \frac{1}{s} = (\text{R.H.S})$$

$$\left[\because \left[\frac{t}{e^{st}} \right]_0^\infty = \frac{1}{e^\infty} - \frac{1}{e^0} \right]$$

$$= \frac{1}{\infty} - \frac{1}{1}$$

$$= 0 - 1 = -1$$

Proof (ii) :-

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

L.H.S $L\{t\}$

$$= \int_0^\infty e^{-st} \cdot t \, dt$$

$$= \left[t \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^\infty$$

$$= \left[0 - \frac{1}{s^2} (0 - 1) \right]$$

$$= \frac{1}{s^2} = \text{R.H.S (proved)}$$

Proof (iii) :-

$$L\{e^{at}\} = \frac{1}{s-a}$$

L.H.S $L\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at} \, dt$

$$= \int_0^\infty e^{-st+at} \, dt = \int_0^\infty e^{-(s-a)t} \, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty = \frac{1}{-(s-a)} [0 - 1]$$

$$= \frac{1}{s-a} = \text{R.H.S}$$

Properties of Laplace Transformation

(1) Property of Linearity :-

$$\begin{aligned} &L\{af(t) + bq(t) - ch(t)\} \\ &= aL\{f(t)\} + bL\{g(t)\} - cL\{h(t)\} \end{aligned}$$

Q:-1 find the Laplace Transforms of .

(i) $e^{2t} + 4t^4 - 3\cos 3t$

(ii) $(\sin t - \cos t)^2$

(iii) $\sin t \cdot \cos 2t$

(iv) $\cos^2 3t$

Solution :-

(i) $L\{e^{2t} + 4t^4 - 3\cos 3t\}$

$$= L\{e^{2t}\} + 4L\{t^4\} - 3L\{\cos 3t\}$$

$$= \frac{1}{s-2} + 4 \frac{4!}{s^5} - 3 \frac{s}{s^2 + (3)^2}$$

$$= \frac{1}{s-2} + 4 \frac{24}{s^5} - \frac{3s}{s^2+9}$$

$$(ii) L\{\sin t - \cos t\}^2$$

$$= L\{\sin^2 t + \cos^2 t - 2 \sin t \cdot \cos t\}$$

$$= L\{1 - 2 \sin t \cdot \cos t\}$$

$$\left(\begin{array}{l} \because 2 \sin A \cdot \cos B \\ = \sin(A+B) + \sin(A-B) \end{array} \right)$$

$$= L\{1\} - 2 L\{\sin t \cdot \cos t\}$$

$$= \frac{1}{s} - 2 L\left\{\frac{1}{2}(\sin(2t) + \sin(0))\right\}$$

$$= \frac{1}{s} - 2 \times \frac{1}{2} L\{\sin 2t\}$$

$$= \frac{1}{s} - \frac{2}{s^2 + (2)^2}$$

$$= \frac{1}{s} - \frac{2}{s^2 + 4} = \frac{s^2 + 4 - 2s}{s(s^2 + 4)}$$

$$(iii) L\{\sin t \cdot \cos 2t\}$$

$$= L\left\{\frac{1}{2}(2 \sin t \cdot \cos 2t)\right\}$$

$$= L\left\{\frac{1}{2}(\sin 3t + \sin(-t))\right\}$$

$$= L\left\{\frac{1}{2}(\sin 3t - \sin t)\right\}$$

$$= \frac{1}{2} \left[L \{ \sin 3t \} - L \{ \sin t \} \right]$$

$$= \frac{1}{2} \left(\frac{3}{s^2+9} - \frac{1}{s^2+1} \right)$$

$$(iv) L \{ \cos^2 3t \}$$

$$\left(\because 2 \cos^2 \theta = 1 + \cos 2\theta \right)$$

$$= L \left\{ \frac{1}{2} (2 \cos^2 3t) \right\}$$

$$= L \left\{ \frac{1}{2} (1 + \cos 6t) \right\}$$

$$= \frac{1}{2} \left[L \{ 1 \} + L \{ \cos 6t \} \right]$$

$$= \frac{1}{2} \left(\frac{1}{s} + \frac{s}{s^2+36} \right)$$

(2) ~~First~~ First shifting property :-

$$L \{ e^{at} f(t) \} = F(s-a)$$

So by applying this Property, we will get the following formulas.

$$(1) L(e^{at}) = \frac{1}{s-a}$$

$$(2) L(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}}$$

$$(3) L(e^{at} \sin bt) = \frac{b}{(s-a)^2 + b^2}$$

$$(4) L(e^{at} \cos bt) = \frac{s-a}{(s-a)^2 + b^2}$$

$$(5) L(e^{at} \sinh bt) = \frac{b}{(s-a)^2 - b^2}$$

$$(6) L(e^{at} \cosh bt) = \frac{(s-a)}{(s-a)^2 - b^2}$$

Q.2 Evaluate the Laplace transforms of

(i) $e^{3t} \cos 4t$

(ii) $e^{2t} t^2$

(iii) $e^{-t} \sin^2 t$

(iv) $e^{-2t} \sinh 3t$

Solutions

(i) $L\{e^{3t} \cos 4t\}$

~~$= \frac{2s}{s^2+16}$~~

Then by first shifting property
 $L\{\cos 4t\} = \frac{s}{s^2+(4)^2}$
 $L\{e^{3t} \cos 4t\} = \frac{(s-3)}{(s-3)^2+(4)^2}$

$$= \frac{(s-3)}{s^2+9-6s+16}$$

$$= \frac{s-3}{s^2-6s+25}$$

(ii) $L\{e^{2t} t^2\}$

$$L\{t^2\} = \frac{2!}{s^2+1} = \frac{2}{s^3}$$

Then by first shifting property,

$$L\{e^{2t} t^2\} = \frac{2}{(s-2)^3}$$

$$(ii) L\{e^{-t} \sin^2 t\}$$

$$L\{\sin^2 t\}$$

$$= L\left\{\frac{1}{2}(2\sin^2 t)\right\}$$

$$= L\left\{\frac{1}{2}(1 - \cos 2t)\right\}$$

$$= L\left\{\frac{1}{2}(1 - \cos 2t)\right\}$$

$$= \frac{1}{2} [L\{1\} - L\{\cos 2t\}]$$

$$= \frac{1}{2} \left[\frac{1}{s} - \frac{s}{s^2+4} \right]$$

Then by first shifting property.

$$L\{e^{-t} \sin^2 t\}$$

$$= \frac{1}{2} \left[\frac{1}{s+1} - \frac{(s+1)}{(s+1)^2+4} \right] \text{ (Ans) .}$$

$$= \frac{1}{2} \left[\frac{(s^2+2s+5) - (s+1)}{(s+1)(s^2+2s+5)} \right]$$

(Ans) .

* Transform of a function multiplied by t^n .

$$\text{i.e. } L \{ t^n f(t) \} = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$\text{Where } L \{ f(t) \} = F(s)$$

Q:-1 Evaluate L.T. of followings.

$$(i) L \{ t \cos at \}$$

Solution

$$L \{ \cos at \} = \frac{s}{s^2 + a^2}$$

$$L \{ t \cos at \} = (-1)^1 \frac{d}{ds} \left(\frac{s}{s^2 + a^2} \right)$$

$$= (-1) \left[\frac{(s^2 + a^2) \frac{d}{ds} s - s \frac{d}{ds} (s^2 + a^2)}{(s^2 + a^2)^2} \right]$$

$$= (-1) \left[\frac{(s^2 + a^2) - s(2s)}{(s^2 + a^2)^2} \right]$$

$$= (-1) \left[\frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2} \right]$$

$$= (-1) \left[\frac{a^2 - s^2}{(s^2 + a^2)^2} \right] = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

$$(ii) \quad L \{ t^2 e^{-3t} \}$$

Solution $L \{ e^{-3t} \}$

$$= \frac{1}{s+3}$$

$$L \{ t^2 e^{-3t} \}$$

$$= (-1)^2 \frac{d^2}{ds^2} \left(\frac{1}{s+3} \right)$$

$$= \frac{d}{ds} \left(\frac{d}{ds} \frac{1}{s+3} \right)$$

$$= \frac{d}{ds} \left(-\frac{1}{(s+3)^2} \right)$$

$$= \frac{-(-2)}{(s+3)^3}$$

$$= \frac{2}{(s+3)^3} \quad (\text{Ans}).$$

OR $L \{ t^2 \}$

$$= \frac{2!}{s^{2+1}}$$

$$= \frac{2}{s^3}$$

Then $L \{ e^{-3t} t^2 \}$

$$= \frac{2}{(s+3)^3} \quad (\text{Ans})$$

$$(iii) \mathcal{L}\{e^{-t} \sin 3t\}$$

$$\underline{\text{solution:}} - \mathcal{L}\{\sin 3t\} = \frac{3}{s^2 + a^2} = \frac{3}{s^2 + 3^2} = \frac{3}{s^2 + 9}$$

$$\mathcal{L}\{t \sin 3t\} = (-1)^1 \frac{d}{ds} \left(\frac{3}{s^2 + 9} \right)$$

$$= (-1) \left[\frac{(s^2 + 9) \left(\frac{d}{ds} 3 \right) - 3 \left(\frac{d}{ds} (s^2 + 9) \right)}{(s^2 + 9)^2} \right]$$

$$= (-1) \left[\frac{(s^2 + 9) \cdot 0 - 3(2s)}{(s^2 + 9)^2} \right]$$

$$= (-1) \frac{(-6s)}{(s^2 + 9)^2}$$

$$= \frac{6s}{(s^2 + 9)^2}$$

$$\mathcal{L}\{e^{-t} + \sin 3t\} = \frac{6(s+1)}{[(s+1)^2 + 9]^2}$$

* Transforms of a function divided by t,

$$\text{i.e. } L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s) ds$$

$$\text{where } F(s) = L\{f(t)\}$$

Q:-1 Evaluate L.T. of the followings.

$$(i) L\left\{\frac{1-e^t}{t}\right\}$$

Solution :- $L\{1-e^t\} = \frac{1}{s} - \frac{1}{s-1}$

$$L\left\{\frac{1-e^t}{t}\right\} = \int_s^{\infty} \left(\frac{1}{s} - \frac{1}{s-1}\right) ds.$$

$$= \left[\log s - \log(s-1)\right]_s^{\infty}$$

$$= \left[\log\left(\frac{s}{s-1}\right)\right]_s^{\infty}$$

$$= \left[\log\left(\frac{1}{1-\frac{1}{s}}\right)\right]_s^{\infty}$$

$$= \log 1 - \log\left(\frac{s}{s-1}\right)$$

$$= 0 - \log\left(\frac{s}{s-1}\right) = \log\left(\frac{s-1}{s}\right) \quad (\text{Ans})$$

Transforms of derivatives

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \dots$$

$$\text{i.e. } L[f'(t)] = s L\{f(t)\} - f(0)$$

$$L\{f''(t)\} = s^2 L\{f(t)\} - s f(0) - f'(0)$$

$$L\{f'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - s f'(0) - f''(0)$$

and so on.

Transforms of Integrals

$$L\left\{\int_0^t f(t) dt\right\} = \frac{F(s)}{s}$$

where $F(s) = L\{f(t)\}$.

Q:-1 Evaluate L.T. of the followings.

$$(1) \int_0^t e^{-3t} \cos t dt :$$

solution $L\{ \cos t \} = \frac{s}{s^2 + 1}$

$$L\{ e^{-3t} \cos t \} = \frac{s+3}{(s+3)^2 + 1}$$

$$L\left\{ \int_0^t e^{-3t} \cos t \, dt \right\} = \frac{s+3}{s[(s+3)^2 + 1]}$$

(ii) $\int_0^t \left(\frac{\sin t}{t} \right) dt$.

solution $L\{ \sin t \} = \frac{1}{s^2 + 1}$

$$L\left\{ \frac{\sin t}{t} \right\} = \int_s^\infty \frac{1}{s^2 + 1} ds.$$

$$= \left[\tan^{-1} s \right]_s^\infty$$

$$= \tan^{-1} \infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$L\left[\int_0^t \left(\frac{\sin t}{t} \right) dt \right] = \frac{\cot^{-1} s}{s} \quad (\text{Ans}).$$

Gamma Function

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx .$$

Properties

$$\textcircled{1} \Gamma(n+1) = n \Gamma(n)$$

or $\Gamma(n+1) = n!$ for n is a positive integer

$$\begin{aligned} \text{Ex: - } \Gamma(4) &= \Gamma(3+1) \\ &= 3 \Gamma(3) \\ &= 3 \Gamma(2+1) \\ &= 3 \cdot 2 \Gamma(2) \\ &= 3 \cdot 2 \Gamma(1+1) \\ &= 3 \cdot 2 \cdot 1 \Gamma(1) \\ &= 3 \cdot 2 \cdot 1 = 3! \end{aligned}$$

Gamma function for $n = \text{fractional number}$,

$$\textcircled{1} \Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2} + 1\right)$$

$$= \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \cdot \sqrt{\pi}$$

NOTE
 $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\textcircled{2} \Gamma\left(\frac{9}{2}\right) = \Gamma\left(\frac{7}{2} + 1\right)$$

$$= \frac{7}{2} \Gamma\left(\frac{7}{2}\right)$$

$$= \frac{7}{2} \cdot \Gamma\left(\frac{5}{2} + 1\right)$$

$$= \frac{7}{2} \cdot \frac{5}{2} \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{7}{2} \cdot \frac{5}{2} \Gamma\left(\frac{3}{2} + 1\right)$$

$$= \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \Gamma\left(\frac{1}{2} + 1\right)$$

$$= \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{105}{16} \sqrt{\pi} \quad (\text{Ans})$$

Q:- Evaluate the followings.

(i) $L\{t^{3/2}\}$

$$\text{Sol}^n \quad L\{t^{3/2}\} = \frac{\Gamma(3/2+1)}{s^{3/2+1}} = \frac{3/2 \Gamma(3/2)}{s^{5/2}}$$

$$= \frac{3/2 \Gamma(1/2+1)}{s^{5/2}} = \frac{3/2 \cdot 1/2 \Gamma(1/2)}{s^{5/2}}$$

$$= \frac{3/4 \sqrt{\pi}}{s^{5/2}} = \frac{3\sqrt{\pi}}{4 s^{5/2}} \quad (\text{Ans})$$

(ii) ~~$L\{t^{1/2}\}$~~ $L\{\sqrt{t}\}$

Solution:- $L\{t^{1/2}\} = \frac{\Gamma(1/2+1)}{s^{1/2+1}} = \frac{1/2 \Gamma(1/2)}{s^{3/2+1}}$

$$= \frac{1/2 \sqrt{\pi}}{s^{3/2}} = \frac{\sqrt{\pi}}{2 s^{3/2}} \quad (\text{Ans})$$

(i) Evaluate: $V(-\frac{1}{2})$

use formula.

Solⁿ

$$V(n+1) = n V(n)$$

$$V(-\frac{1}{2}) = \frac{V(-\frac{1}{2}+1)}{-\frac{1}{2}}$$

$$\Rightarrow \boxed{V(n) = \frac{V(n+1)}{n}}$$

$$= 2 V(\frac{1}{2})$$

$$= -2\sqrt{\pi}$$

$$\text{as } \boxed{V(\frac{1}{2}) = \sqrt{\pi}}$$

(ii) Evaluate $V(-\frac{7}{2})$

$$\text{Solⁿ :- } V(-\frac{7}{2}) = \frac{V(-\frac{7}{2}+1)}{-\frac{7}{2}}$$

$$= \frac{V(-\frac{5}{2})}{-\frac{7}{2}}$$

$$= \frac{\cancel{V(-\frac{5}{2}+1)}}{-\frac{7}{2} \cdot -\frac{5}{2}}$$

$$= -\frac{2}{7} \frac{V(-\frac{5}{2}+1)}{-\frac{5}{2}}$$

$$= -\frac{2}{7} \cdot (-\frac{2}{5}) V(-\frac{3}{2})$$

$$= \frac{4}{35} \frac{V(-\frac{3}{2}+1)}{-\frac{3}{2}}$$

$$= \frac{4}{35} \left(-\frac{2}{3}\right) V\left(-\frac{1}{2}\right)$$

$$= \frac{4}{35} \left(-\frac{2}{3}\right) (-2\sqrt{\pi})$$

$$= \frac{8 \cdot 2}{105} \sqrt{\pi} = \frac{16}{105} \sqrt{\pi}$$

(iii) Evaluate. ~~L~~ $L\left(\frac{1}{\sqrt{t}}\right)$

$$\underline{\text{Sol}^n} \quad L\left(\frac{1}{\sqrt{t}}\right)$$

$$= L\left(t^{-\frac{1}{2}}\right)$$

$$= \left\{ \frac{V(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}} \right\}$$

$$= \frac{V\left(\frac{1}{2}\right)}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} \quad (\text{Ans})$$

Inverse Laplace Transformation:-

Inverse Laplace Transformation is the inverse function of Laplace Transformation.

$$\text{i.e. } L\{f(t)\} = F(s)$$

$$\text{Then } \boxed{f(t) = L^{-1} F(s)}$$

Then we have the following results from the Laplace Transforms:-

$$\textcircled{1} L^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\textcircled{2} L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$\textcircled{3} L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

$$\textcircled{4} L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$$

$$\text{or } \boxed{L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}}$$

$$\textcircled{5} \quad L^{-1} \left\{ \frac{s}{s^2 + a^2} \right\} = \cos at$$

$$\textcircled{6} \quad L^{-1} \left\{ \frac{s}{s^2 - a^2} \right\} = \cosh at.$$

$$\textcircled{7} \quad L^{-1} \left\{ \frac{a}{s^2 - a^2} \right\} = \sinh at$$

$$\Rightarrow L^{-1} \left\{ \frac{1}{s^2 - a^2} \right\} = \frac{1}{a} \sinh at$$

$$\textcircled{8} \quad L^{-1} \left\{ \frac{a}{s^2 + a^2} \right\} = \sin at.$$

$$\Rightarrow L^{-1} \left\{ \frac{1}{s^2 + a^2} \right\} = \frac{1}{a} \sin at$$

$$\textcircled{9} \quad L^{-1} \left\{ \frac{s+b}{(s+b)^2 + a^2} \right\} = e^{-bt} \cos at$$

$$\textcircled{10} \quad L^{-1} \left\{ \frac{1}{(s+b)^2 + a^2} \right\} = \frac{1}{a} e^{-bt} \sin at.$$

$$\textcircled{11} \quad L^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\} = \frac{1}{2a} t \sin at.$$

$$\textcircled{12} \quad L^{-1} \left\{ \frac{1}{(s^2 + a^2)^2} \right\} = \frac{1}{2a^3} (\sin at - at \cos at)$$

Q:-1 Evaluate inverse Laplace Transform of the following functions:

$$(i) \frac{2s}{s^2+9}$$

$$(ii) \frac{3s-4}{s^2-25}$$

$$(iii) \frac{1}{s^3}$$

$$(iv) \frac{1}{(s+5)^3}$$

$$(v) \frac{25}{s-6}$$

Solution:-

$$(i) L^{-1} \left\{ \frac{2s}{s^2+9} \right\}$$

$$= 2 L^{-1} \left\{ \frac{s}{s^2+9} \right\}$$

$$= 2 \cos 3t$$

$$(ii) L^{-1} \left\{ \frac{3s-4}{s^2-25} \right\}$$

$$= L^{-1} \left\{ \frac{3s}{s^2-25} \right\} - 4 L^{-1} \left\{ \frac{1}{s^2-25} \right\}$$

$$= 3 \cosh 5t - 4 \cdot \frac{1}{5} \sinh 5t$$

$$(iii) \mathcal{L}^{-1} \left\{ \frac{1}{s^3} \right\}$$

$$= \frac{(3-1)!}{t^{3-1}}$$

$$\frac{t^{3-1}}{(3-1)!}$$

$$= \frac{2!}{t^2} = \frac{2}{t^2}$$

$$= \frac{t^2}{2!}$$

$$(iv) \mathcal{L}^{-1} \left\{ \frac{1}{(s+5)^3} \right\}$$

$$= e^{-5t} \cdot \frac{t^2}{2!}$$

$$(v) \mathcal{L}^{-1} \left\{ \frac{25}{s-6} \right\}$$

$$= 25 \mathcal{L}^{-1} \left\{ \frac{1}{s-6} \right\}$$

$$= 25 e^{6t}$$

Partial Fraction :-

Case I :- when denominator contains non-repeated linear factors.

$$\frac{P(x)}{(x-a)(x+b)(x-c)} = \frac{A}{x-a} + \frac{B}{x+b} + \frac{C}{x-c}$$

Case II :- when denominator contains repeated linear factors.

$$\frac{P(x)}{(x-a)^2(x+b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x+b}$$

Case III :- when denominator contains non-repeated quadratic factors which can't be factorised.

$$\frac{P(x)}{(x^2+a)(x^2+c)} = \frac{Ax+B}{x^2+a} + \frac{Cx+D}{x^2+c}$$

If the quadratic term is factorised then it converted into the linear factors.

In some cases it is not possible to calculate inverse L.T. as it is not in standard form, In that case $f(t)$ can be obtained by resolving $F(s)$ into partial fraction.

Q.1 $L^{-1} \left(\frac{s}{(s-3)(s^2+4)} \right)$

Solⁿ consider

$$\Rightarrow \frac{s}{(s-3)(s^2+4)} = \frac{A}{s-3} + \frac{Bs+C}{s^2+4} \quad \text{--- (1)}$$

$$= \frac{A(s^2+4) + (Bs+C)(s-3)}{(s-3)(s^2+4)}$$

$$\Rightarrow s = A(s^2+4) + (Bs+C)(s-3)$$

$$\Rightarrow s = As^2 + 4A + Bs^2 - 3Bs + Cs - 3C$$

$$= (A+B)s^2 + (C-3B)s + 4A-3C$$

Comparing the coefficients of s^2 , s and constants.

$$A+B=0 \quad \text{--- (1)}$$

$$C-3B=1 \quad \text{--- (11)}$$

$$4A-3C=0 \quad \text{--- (10)}$$

we have

$$A+B=0$$

$$\Rightarrow A=-B$$

put in eqⁿ (11)

$$C-3(-A)=1$$

$$\Rightarrow C+3A=1$$

Now

$$3A+C=1$$

$$4A-3C=0$$

Solving these two eqⁿs.

$$A = \frac{3}{13}, \quad C = \frac{4}{13}$$

$$\text{and } B=-A \Rightarrow B = -\frac{3}{13}$$

putting the values in eqⁿ (1)

$$\Rightarrow \frac{s}{(s-3)(s^2+4)} = \frac{\frac{3}{13}}{s-3} + \frac{(-\frac{3}{13})s + \frac{4}{13}}{s^2+4}$$

Taking inverse L.T. on both sides.

$$\Rightarrow \mathcal{L}^{-1} \left\{ \frac{s}{(s-3)(s^2+4)} \right\} = \mathcal{L}^{-1} \left\{ \frac{3/13}{s-3} \right\} \\ + \mathcal{L}^{-1} \left\{ \frac{-3/13 s}{s^2+4} \right\} + \mathcal{L}^{-1} \left\{ \frac{4/13}{s^2+4} \right\} \\ = \frac{3}{13} \mathcal{L}^{-1} \left\{ \frac{1}{s-3} \right\} - \frac{3}{13} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+4} \right\} \\ + \frac{4}{13} \mathcal{L}^{-1} \left\{ \frac{1}{s^2+4} \right\}$$

$$= \frac{3}{13} e^{3t} - \frac{3}{13} \cos 2t + \frac{4}{13} \cdot \frac{1}{2} \sin 2t$$

(Ans).

Q.2 $\mathcal{L}^{-1} \left\{ \frac{3s}{s^2+2s-8} \right\}$

Solⁿ $\mathcal{L}^{-1} \left\{ \frac{3s}{s^2+4s-2s-8} \right\}$

$$= \mathcal{L}^{-1} \left\{ \frac{3s}{s(s+4)-2(s+4)} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{3s}{(s+4)(s-2)} \right\}$$

Now.

$$\left\{ \frac{3s}{(s+4)(s-2)} \right\} = \frac{A}{s+4} + \frac{B}{s-2} \quad \text{--- (i)}$$
$$= \frac{A(s-2) + B(s+4)}{(s+4)(s-2)}$$

$$\Rightarrow 3s = A(s-2) + B(s+4)$$
$$= As - 2A + Bs + 4B$$
$$= (A+B)s + 4B - 2A$$

Comparing the coefficients of s and constants.

$$A+B=3 \quad \text{--- (ii)}$$

$$4B-2A=0 \quad \text{--- (iii)}$$

Solving (ii) & (iii)

$$2A + 2B = 6$$
$$-2A + 4B = 0$$

$$6B = 6$$

$$\boxed{B=1}$$

$$\text{Then } \boxed{A=2}$$

Putting these values in eqⁿ (i)

$$\Rightarrow \frac{3s}{(s+4)(s-2)} = \frac{2}{s+4} + \frac{1}{s-2}$$

Taking I.L.T. on both sides.

$$\Rightarrow \mathcal{L}^{-1} \left\{ \frac{3s}{s^2 + 2s - 8} \right\} = \mathcal{L}^{-1} \left\{ \frac{2}{s+4} \right\} + \mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\}$$
$$= 2e^{-4t} + e^{2t} \quad (\text{Ans})$$

$$\underline{\text{Q.3}} \quad \mathcal{L}^{-1} \left\{ \frac{s+2}{s^2-4s+13} \right\}$$

$$\text{consider } \frac{s+2}{s^2-4s+13}$$

$$= \frac{s+2}{s^2-2 \cdot s \cdot 2 + (2)^2 - (2)^2 + 13}$$

$$= \frac{s+2}{(s-2)^2 + 9}$$

$$= \frac{s-2+4}{(s-2)^2+9}$$

$$\frac{s+2}{s^2-4s+13} = \frac{s-2}{(s-2)^2+9} + \frac{4}{(s-2)^2+9}$$

Taking J.L.T. on both sides.

$$\Rightarrow \mathcal{L}^{-1} \left\{ \frac{s+2}{s^2-4s+13} \right\} = \mathcal{L}^{-1} \left\{ \frac{s-2}{(s-2)^2+9} \right\} + 4 \mathcal{L}^{-1} \left\{ \frac{1}{(s-2)^2+9} \right\}$$

$$= e^{2t} \cos 3t + 4 \cdot \frac{1}{3} e^{2t} \sin 3t.$$

[as we can't
break the
middle term
So make perfect
square]

Solution of linear differential eqⁿs using L.T.

Q.1: - Solve $y'' + 5y' + 6y = e^{2x}$, $y(0) = 2$, $y'(0) = -1$

Solution :- Given $y'' + 5y' + 6y = e^{2x}$

Taking L.T. on both sides -

$$L\{y''(x) + 5y'(x) + 6y(x)\} = L\{e^{2x}\}$$

$$\Rightarrow L\{y''(x)\} + 5L\{y'(x)\} + 6L\{y(x)\} = \frac{1}{s-2}$$

$$\Rightarrow [s^2 L\{y(x)\} - sy(0) - y'(0)] + 5[sL\{y(x)\} - y(0)] + 6L\{y(x)\} = \frac{1}{s-2}$$

$$\Rightarrow s^2 L\{y(x)\} - 2s - (-1) + 5sL\{y(x)\} - 5(-1) + 6L\{y(x)\} = \frac{1}{s-2}$$

$$\Rightarrow \{s^2 + 5s + 6\} L\{y(x)\} - 2s + 1 + 5 = \frac{1}{s-2}$$

$$\Rightarrow (s^2 + 5s + 6) L\{y(x)\} = \frac{1}{s-2} + 2s - 6$$

$$\Rightarrow (s^2 + 5s + 6) \mathcal{L}\{y(x)\} = \frac{1 + (2s-6)(s-2)}{s-2}$$

$$= \frac{1 + 2s^2 - 4s - 6s + 12}{s-2}$$

$$\Rightarrow \mathcal{L}\{y(x)\} = \frac{2s^2 - 10s + 13}{(s-2)(s^2 + 5s + 6)}$$

$$= \frac{2s^2 - 10s + 13}{(s-2)(s+2)(s+3)}$$

$$\Rightarrow y(x) = \mathcal{L}^{-1}\left\{\frac{2s^2 - 10s + 13}{(s-2)(s+2)(s+3)}\right\}$$

Consider $\frac{2s^2 - 10s + 13}{(s-2)(s+2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s+3}$ — (1)

$$= \frac{A(s+2)(s+3) + B(s-2)(s+3) + C(s-2)(s+2)}{(s-2)(s+2)(s+3)}$$

$$\Rightarrow 2s^2 - 10s + 13 = A(s+2)(s+3) + B(s-2)(s+3) + C(s-2)(s+2)$$
 — (11)

Expand eqⁿ (11)

$$= A(s^2 + 3s + 2s + 6) + B(s^2 + 3s - 2s - 6) + C(s^2 - 4)$$

$$= As^2 + 5As + 6A + Bs^2 + Bs - 6B + Cs^2 - 4C$$

$$= (A+B+C)s^2 + (5A+B)s + 6A - 6B - 4C$$

Comparing both sides.

$$A+B+c=2 \quad \text{--- (i)}$$

$$5A+B=-10 \quad \text{(ii)} \Rightarrow B=-10-5A$$

$$6A-6B-4c=13 \quad \text{--- (iii)}$$

putting the values in eqⁿ (i) & (iii)

$$A-10-5A+c=2 \Rightarrow -4A+c=12 \quad \text{--- (iv)}$$

$$6A-6(-10-5A)-4c=13 \Rightarrow 6A+60+30A-4c=13$$
$$\Rightarrow 36A-4c=-47 \quad \text{--- (v)}$$

Solving (iv) & (v)

$$\begin{array}{r} -16A+4c=48 \\ 36A-4c=-47 \\ \hline \end{array}$$

$$20A=1$$

$$\boxed{A = \frac{1}{20}}$$

$$c = \frac{61}{5}$$

$$B = \frac{-41}{4}$$

putting in eqⁿ (i)

$$\Rightarrow \frac{2s^2 - 10s + 13}{(s-2)(s+2)(s+3)} = \frac{\frac{1}{20}}{s-2} + \frac{\left(\frac{-41}{4}\right)}{s+2} + \frac{\frac{61}{5}}{s+3}$$

Taking Inverse L.T. on both sides,

$$\Rightarrow \mathcal{L}^{-1} \left\{ \frac{2s^2 - 10s + 13}{(s-2)(s+2)(s+3)} \right\} = \frac{1}{20} \mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\} - \frac{41}{4} \mathcal{L}^{-1} \left\{ \frac{1}{s+2} \right\} + \frac{61}{5} \mathcal{L}^{-1} \left\{ \frac{1}{s+3} \right\}$$

$$= \frac{1}{20} e^{2t} - \frac{41}{4} e^{-2t} + \frac{61}{5} e^{-3t}.$$

(Ans)

Fourier Series :-

Defⁿ :- Let $f(x)$ be a periodic function with period 2π . Then the trigonometric series.

$$\text{i.e. } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

is called Fourier series where a_0, a_n, b_n are the Fourier coefficients of $f(x)$.

Periodic function

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is said to be a periodic function of period λ (a +ve no.) if

$$f(x + \lambda) = f(x)$$

Example $f(x) = \sin x$.

$$\text{Now } \sin(x + 2\pi) = \sin x$$

So $\sin x$ is a periodic function of period 2π .

* Fourier Series is represented in the form of Euler's formulae.

Euler's formulae

The Fourier Series for the function $f(x)$ in the interval $(\alpha, \alpha+2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

where

$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx.$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx.$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx dx.$$

Now these values a_0, a_n, b_n are known as Euler's formulae.

NOTE-1

① If $\alpha = 0$ then ~~interval~~ interval becomes $(0, 2\pi)$

Then

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx.$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx.$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx.$$

② If $\alpha = -\pi$, then the interval becomes $(-\pi, \pi)$.

Then

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx. \quad \text{or } a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad \text{or } a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad \text{or } b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx.$$

NOTE-2 If the given interval is $(-\pi, \pi)$

then check

① If $f(x)$ is an even function, then $b_n = 0$

$$* a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx, \quad a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \, dx$$

② If $f(x)$ is an odd function, then $a_0 = 0$

$$* a_n = 0$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

Q.1 Find Fourier expansion of $f(x) = x$
 $-\pi < x < \pi$

Solⁿ Here $f(x) = x$ is an odd function, so

$$a_0 = 0 \quad * \quad a_n = 0$$

$$* b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi} \left[x \left(\frac{-\cos nx}{n} \right) - 1 \left(\frac{-\sin nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{-x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left\{ \frac{-\pi \cos n\pi}{n} + \frac{\sin n\pi}{n^2} \right\} - \left\{ \frac{0}{n} + \frac{\sin 0}{n^2} \right\} \right]$$

$$= \frac{2}{\pi} \left[\left\{ \frac{(-1)^n}{n} + 0 \right\} - 0 \right] \int u v dx = u \int v dx - \left(\frac{d}{dx} u \right) \int \int v dx - \left(\frac{d^2}{dx^2} u \right) \int \int \int v dx \dots$$

$$\frac{2}{\pi} \int \frac{2}{n} (-1)^{n+1}$$

Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx$$

$$= 2 \left[\sin x + \left(\frac{-1}{2} \right) \sin 2x + \frac{1}{3} \sin 3x - \dots \right]$$

(Ans).