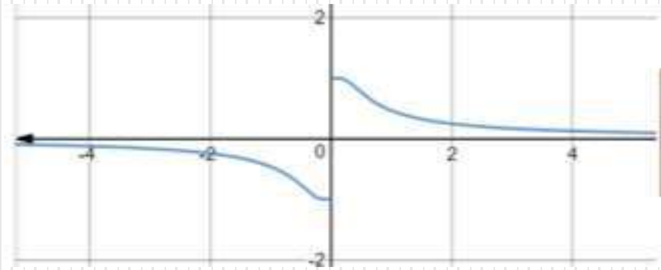
$$f(x) = \begin{cases} x, & 0 \leqslant x < \frac{1}{2} \\ \frac{1}{2}, & x = \frac{1}{2} \\ 1 - x, & \frac{1}{2} < x \leqslant 1 \end{cases} \quad \text{at } x = \frac{1}{2}$$



$$f(x) = \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$$

LIMITS

$$\lim_{x \to a} f(x) = l$$

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LECTURER IN MATHEMATICS

GOVT. POLYTECHNIC JAJPUR

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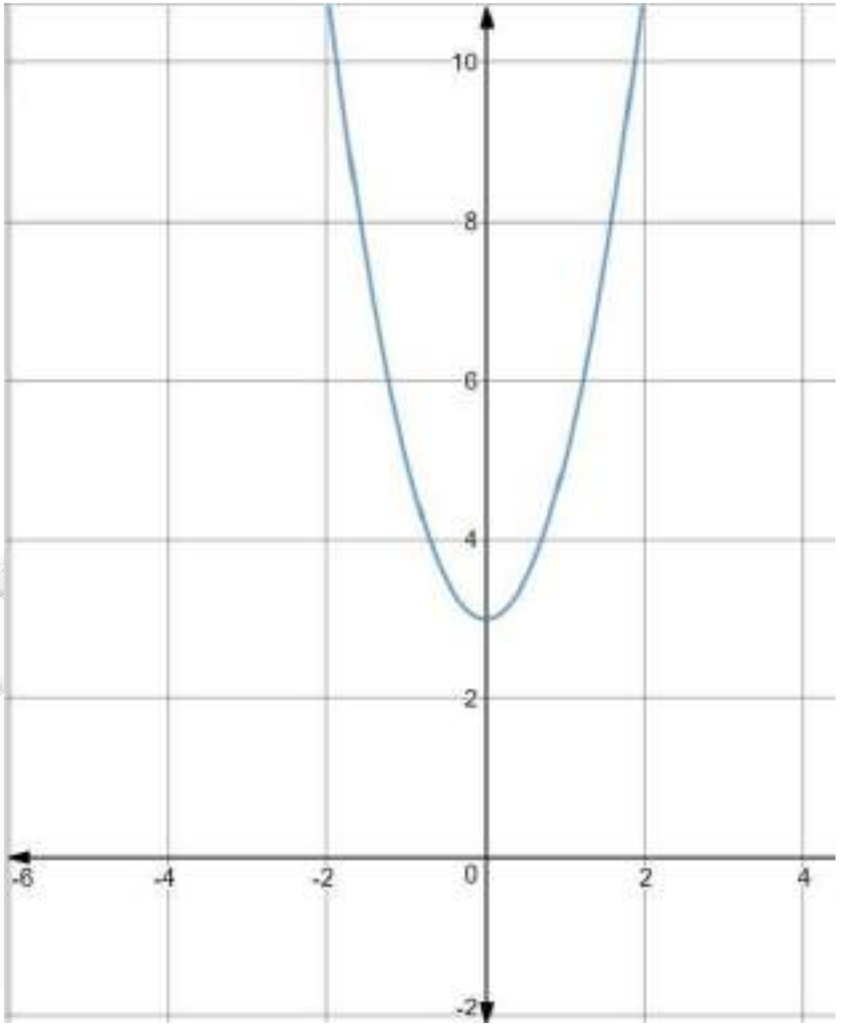
LIMITS

Introduction



Example 1:

let's consider a function i.e. $f(x)$

$$f(x) = 2x^2 + 3$$

at $x = 2$, $\quad f(2) = 2(2)^2 + 3 = 2(4) + 3 = 11$

at $x = -1$, $f(-1) = 2(-1)^2 + 3 = 2(1) + 3 = 5$

The function $f(x) = 2x^2 + 3$ is defined for all $x \in \mathbb{R}$

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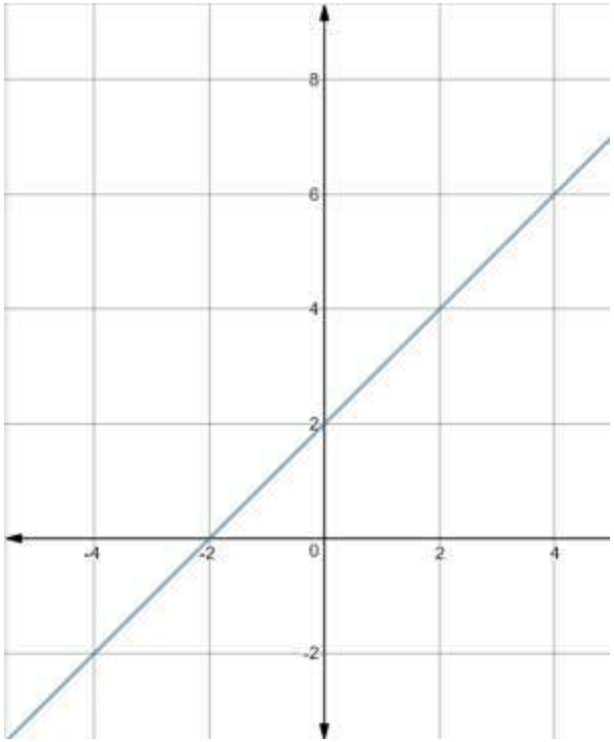
Example 2:

Lets consider another function

$$f(x) = \frac{x^2 - 4}{x - 2}$$



at $x = 1$, $f(1) = \frac{(1)^2 - 4}{1 - 2} = \frac{-3}{-1} = 3$

at $x = -1$, $f(-1) = \frac{(-1)^2 - 4}{-1 - 2} = \frac{-3}{-3} = 1$

at $x = 2$, $f(2) = \frac{(2)^2 - 4}{2 - 2} = \frac{0}{0}$ (indeterminate form)

So clearly this function is defined for all x except 2.

Although $f(x) = \frac{x^2 - 4}{x - 2}$ is not defined at $x = 2$

(i.e. its functional value at $x = 2$ doesn't exist)

But we can study how this function behaves in the neighbourhood of $x = 2$

by using the concept of LIMIT.

Video links

The following table shows how the function behaves when we come closer to 2. from both left hand side **LHS** & right hand side **RHS**

x	1.7	1.8	1.9	1.99	2	2.01	2.1	2.2	2.3
$f(x)$	3.7	3.8	3.9	3.99	$\frac{0}{0}$	4.01	4.1	4.2	4.3

From the above table we observe that

when x comes closer to 2 from $L.H.S.$ $f(x)$ comes closer to 4.

or

when x approaching to 2 from $L.H.S.$ $f(x)$ tends to the limit 4.

or

when $x \to 2^-$, $f(x) \to 4$

or

i.e. $\lim_{x \to 2^-} f(x) = 4$

$\Longrightarrow$ Left hand limit $(L.H.L)$

From the above table we observe that

when x comes closer to 2 from $R.H.S.$ $f(x)$ comes closer to 4 also.

or

when x approaching to 2 from $R.H.S.$ $f(x)$ tends to the limit 4.

or

when $x \to 2^+$, $f(x) \to 4$

or

i.e. $\lim_{x \to 2^+} f(x) = 4$

$\Longrightarrow$ Right hand limit $(R.H.L)$

Here L.H.L = R.H.L

Video links

Example 3:

let's consider a function i.e. $f(x)$

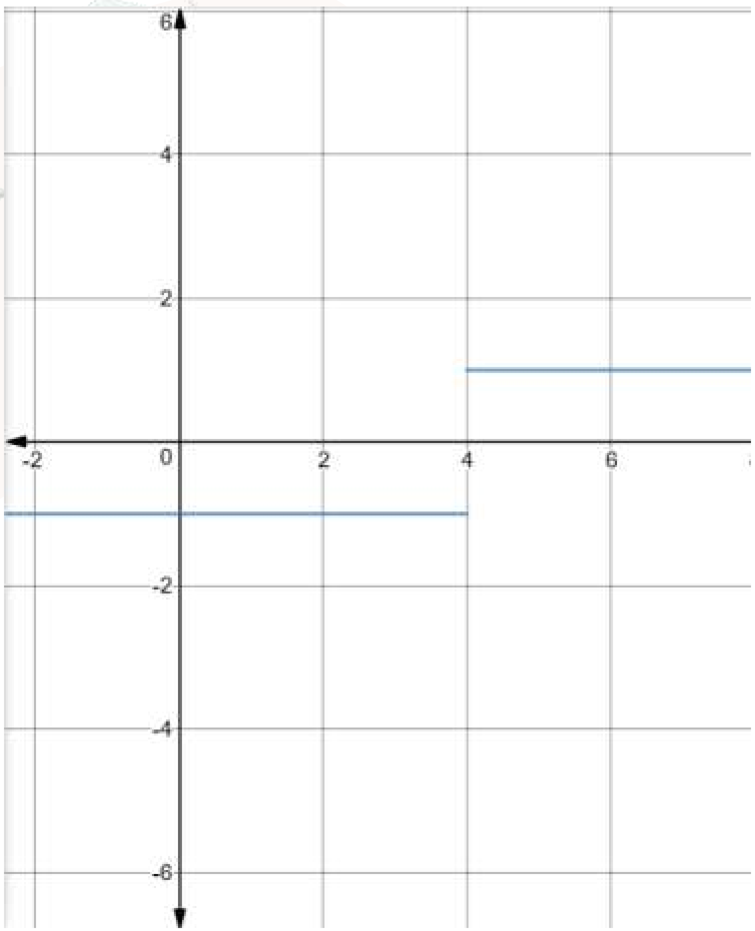
$$f(x) = \frac{|x-4|}{x-4}$$

at $x = 4$, $\quad f(4) = \frac{|x-4|}{x-4} = \frac{0}{0}$

The function $f(x)$ is defined for all $x \in R$ except 4.

So lets check how it behaves in the neighborhood of 4 by taking the help of limit.

x	3.7	3.8	3.9	3.99	4	4.01	4.1	4.2	4.3
$f(x)$	-1	-1	-1	-1	$\frac{0}{0}$	1	1	1	1



$L.H.L \implies \lim_{x \to 4^-} f(x) = -1$

$R.H.L \implies \lim_{x \to 4^+} f(x) = 1$

Here L.H.L ≠ R.H.L

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Example 4

let's consider a function i.e. $f(x)$

$$f(x) = \frac{1}{x-3}$$

at $x = 3, \quad f(3) = \frac{1}{3-3} = \frac{1}{0}$ (undefined form)

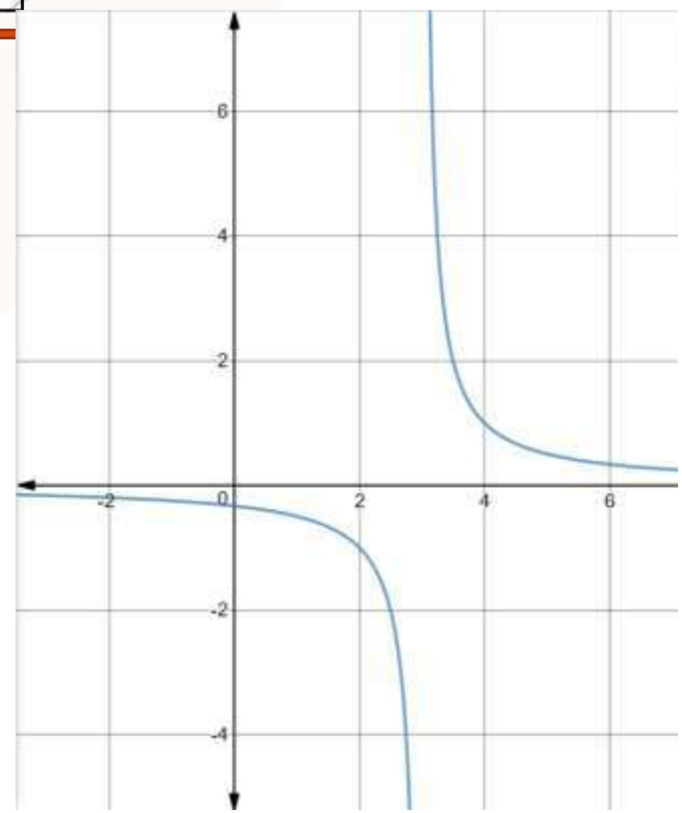
The function $f(x)$ is defined for all $x \in R$ except 3.

So lets check how it behaves in the neighborhood of 3 by taking the help of limit.

x	2.8	2.9	2.99	2.999	3	3.001	3.01	3.1	3.2
$f(x)$	-5	-10	-100	-1000	$\frac{1}{0}$	1000	100	10	5

$L.H.L \implies \lim_{x \to 3^-} f(x) = -\infty$ (doesn't exist)

$R.H.L \implies \lim_{x \to 3^+} f(x) = \infty$(doesn't exist)



Here we can't get any definite number

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Existence of Limit

Note: from the earlier examples (1 to 4) we observe that for some functions

L.H.L = R.H.L (example 2)

L.H.L → Left Hand Limit
R.H.L → Right Hand Limit

L.H.L ≠ R.H.L (example 3)

L.H.L or R.H.L or both not defined (example 4)

THEOREM: **EXISTENCE OF LIMIT**

If **L.H.L = R.H.L**, then we can say limit of the function exists.

Video links

Definition of Limit

- Let *f(x)* be a function defined in neighborhood of '***a***', except '***a***'.
- Let '*l*' be any number.
- Then we can say limit of *f(x)* as '*x*' approaching to '***a***' is '*l*'.

i.e.

$$\lim_{x \to a} f(x) = l$$

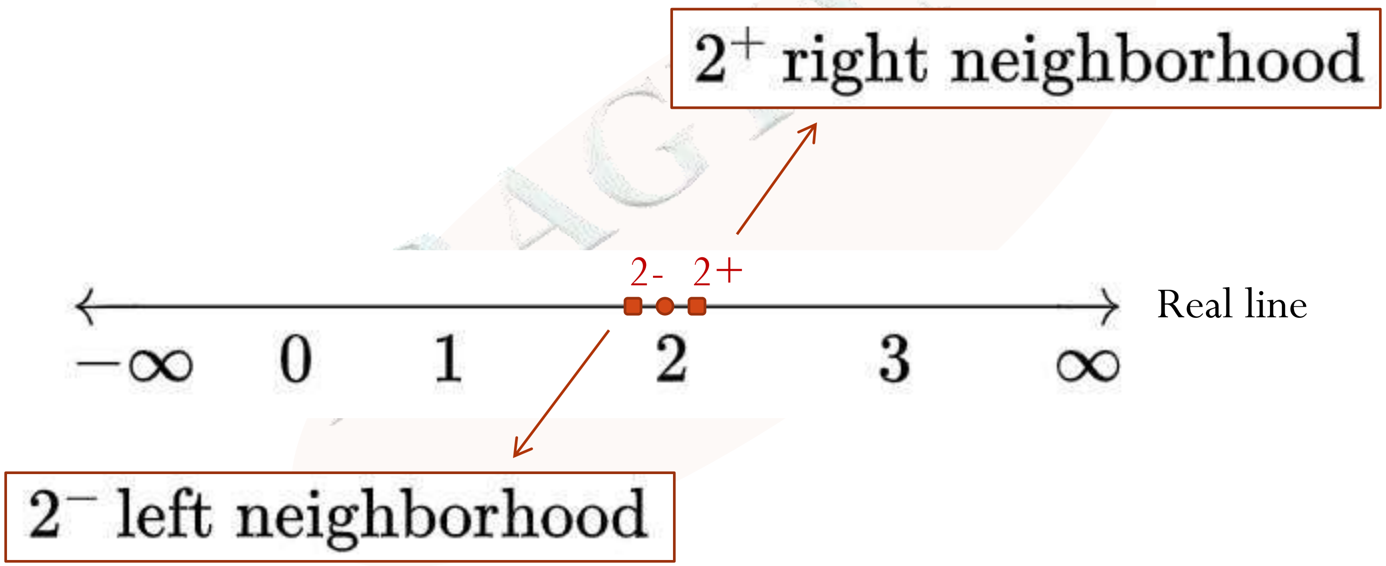
Note:

1. The limit depends upon the values of *f(x)* in the neighborhood of '***a***', except '***a***'.
2. The function *f(x)* may or may not be defined at '***a***'.

Video links

Neighborhood of a point

- Let's check neighborhood of point '**2**'.



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Evaluation of L.H.L and R.H.L

- **LEFT HAND LIMIT**

To evaluate $L.H.L$ of a function $f(x)$ at $x = a$ we have to follow the following steps

step 1: write $\lim_{x \to a^-} f(x)$

step 2: put $x = a - h$

$\left[\text{replace } x \to a^- \text{ by } h \to 0\right]$

$$x \to a^-$$
$$\cancel{a} - h \to \cancel{a}^-$$
$$-h \to 0$$
$$h \to 0$$

$$\lim_{x \to a^-} f(x) \implies \lim_{h \to 0} f(a - h)$$

step 3: simplify $\lim_{h \to 0} f(a - h)$

- **RIGHT HAND LIMIT**

To evaluate $R.H.L$ of a function $f(x)$ at $x = a$ we have to follow the following steps

step 1: write $\lim_{x \to a^+} f(x)$

step 2: put $x = a + h$

$\left[\text{replace } x \to a^+ \text{ by } h \to 0\right]$

$$x \to a^+$$
$$\cancel{a} + h \to \cancel{a}^+$$
$$h \to 0$$

$$\lim_{x \to a^+} f(x) \implies \lim_{h \to 0} f(a + h)$$

step 3: simplify $\lim_{h \to 0} f(a + h)$

Video links

Q1 Evaluate L.H.L and R.H.L where $f(x) = \begin{cases} \frac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases}$ at $x = 4$

- LEFT HAND LIMIT

$$= \lim_{x \to 4^-} f(x)$$

$$= \lim_{x \to 4^-} \frac{|x-4|}{x-4} \{\text{put } x = 4 - h\}$$

$$= \lim_{h \to 0} \frac{|(4-h)-4|}{(4-h)-4}$$

$$= \lim_{h \to 0} \frac{|-h|}{h}$$

$$= \lim_{h \to 0} \frac{h}{-h}$$

$$= \lim_{h \to 0} -1$$

$$= -1$$

- RIGHT HAND LIMIT

$$= \lim_{x \to 4^+} f(x)$$

$$= \lim_{x \to 4^+} \frac{|x-4|}{x-4} \{\text{put } x = 4 + h\}$$

$$= \lim_{h \to 0} \frac{|(4+h)-4|}{(4+h)-4}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= \lim_{h \to 0} 1$$

$$= 1$$

L.H.L ≠ R.H.L → $\lim_{x \to 4} f(x)$ doesn't exist

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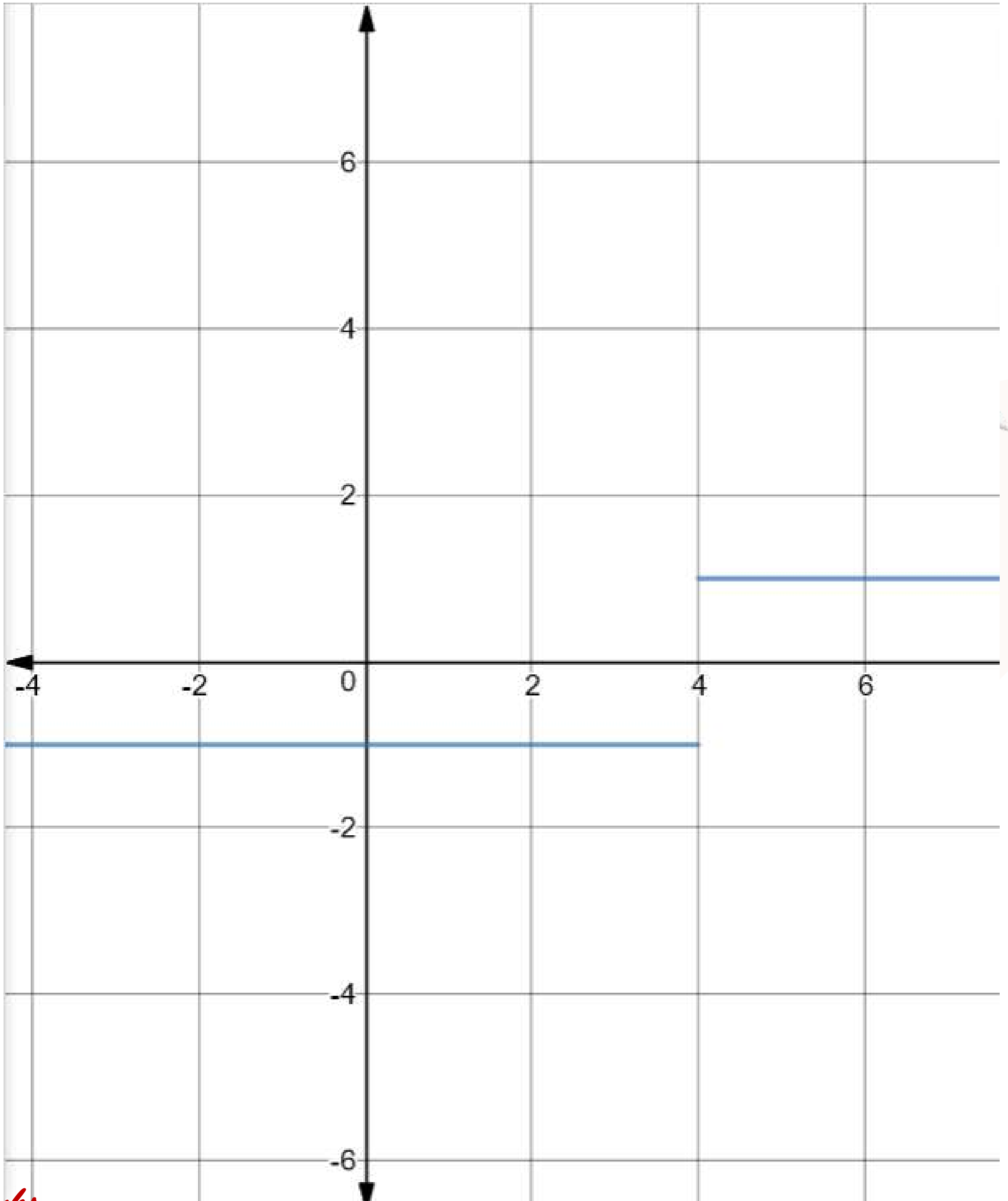

$$f(x) = \frac{|x-4|}{x-4}$$

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Q2

If $f(x) = \begin{cases} \frac{x-|x|}{x}, & x \neq 0 \\ 2, & x = 4 \end{cases}$

check whether $\lim_{x \to 0} f(x)$ exists or not

- LEFT HAND LIMIT

$$= \lim_{x \to 0^-} f(x)$$

$$= \lim_{x \to 0^-} \frac{x-|x|}{x} \{\text{put } x = 0 - h\}$$

$$= \lim_{h \to 0} \frac{-h-|-h|}{-h}$$

$$= \lim_{h \to 0} \frac{-h-h}{-h}$$

$$= \lim_{h \to 0} \frac{-2h}{-h}$$

$$= \lim_{h \to 0} +2$$

$$= 2$$

- RIGHT HAND LIMIT

$$= \lim_{x \to 0^+} f(x)$$

$$= \lim_{x \to 0^+} \frac{x-|x|}{x} \{\text{put } x = 0 + h\}$$

$$= \lim_{h \to 0} \frac{h-|h|}{h}$$

$$= \lim_{h \to 0} \frac{h-h}{h}$$

$$= \lim_{h \to 0} \frac{0}{h}$$

$$= \lim_{h \to 0} 0$$

$$= 0$$

L.H.L ≠ R.H.L → $\lim_{x \to 0} f(x)$ doesn't exist

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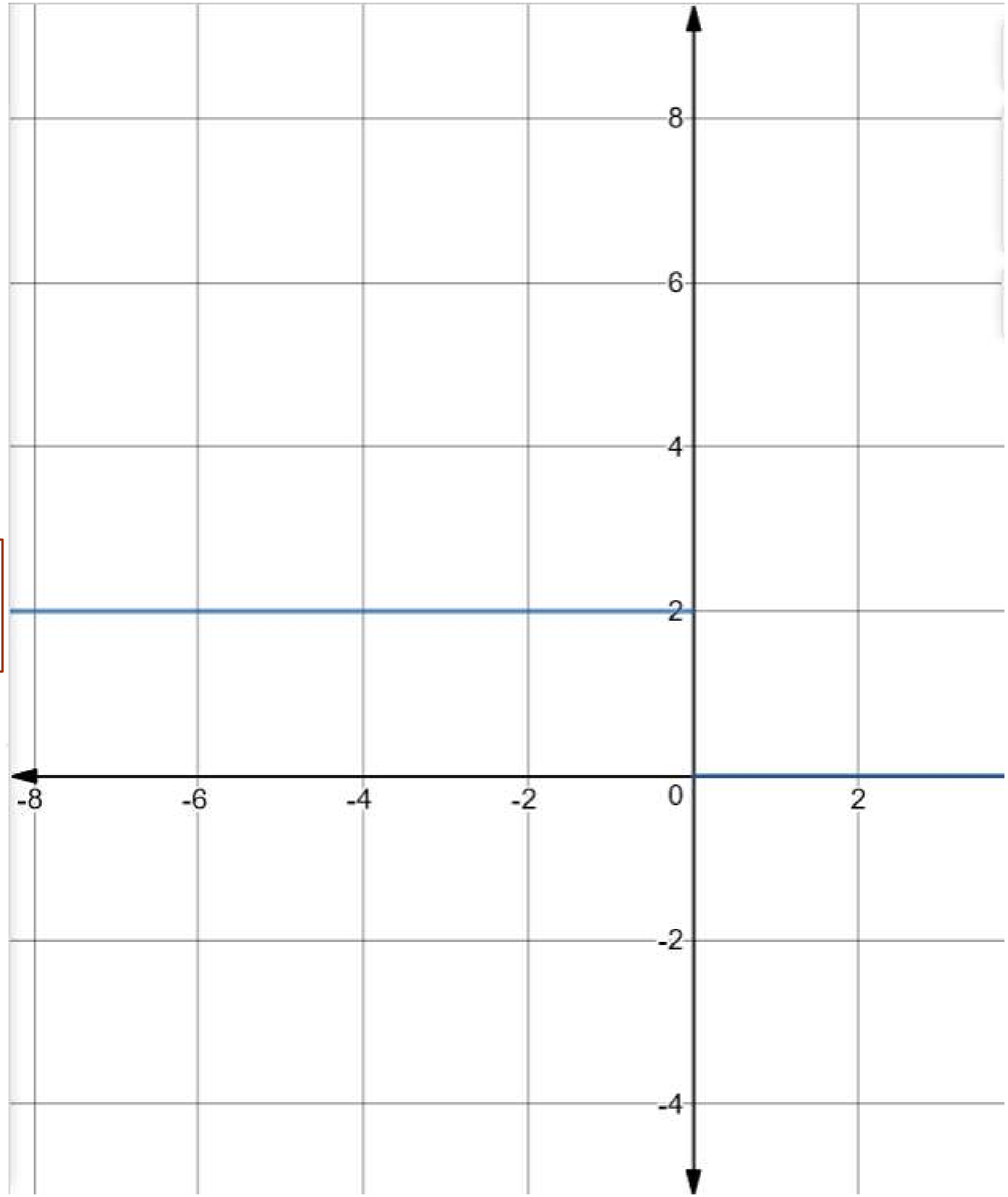
$$f(x) = \frac{x - |x|}{x}$$



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Q3 If $f(x)=\begin{cases}5x-4, & 0<x\leqslant 1\\ 4x^3-3x, & 1<x<2\end{cases}$

show that $\lim_{x\to 1} f(x)$ exists

Note:

1. $f(x)$ at $x=a$ {i.e. functional value of $f(x)$}
2. $f(x)$ at $x\neq a$ {i.e. functional value of $f(x)$}

L.H.L. → $x<a$ R.H.L. → $x>a$

- LEFT HAND LIMIT $(x<a)$

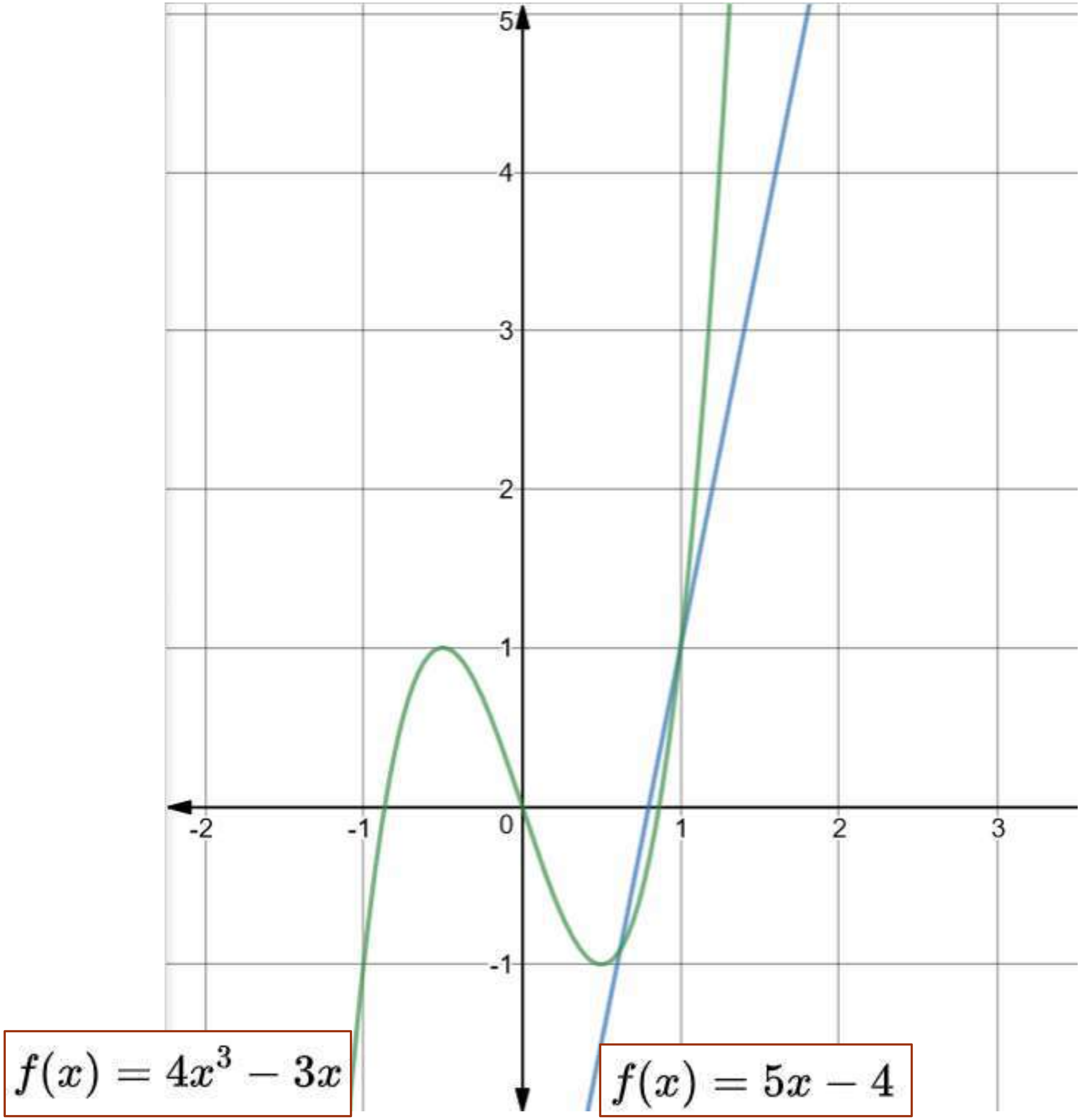
$$=\lim_{x\to 1^-} f(x)$$
$$=\lim_{x\to 1^-} 5x-4 \,\{\text{put } x=1-h\}$$
$$=\lim_{h\to 0} 5(1-h)-4$$
$$=5(1-0)-4$$
$$=5(1)-4$$
$$=5-4$$
$$=1$$

- RIGHT HAND LIMIT $(x>a)$

$$=\lim_{x\to 1^+} f(x)$$
$$=\lim_{x\to 1^+} 4x^3-3x \,\{\text{put } x=1+h\}$$
$$=\lim_{h\to 0} 4(1+h)^3-3(1+h)$$
$$=4(1+0)^3-3(1+0)$$
$$=4(1)^3-3(1)$$
$$=4-3$$
$$=1$$

L.H.L = R.H.L → $\lim_{x\to 1} f(x)=1$ exists

Video links



Video links

Q4 Examine the existence of the function

If $f(x)=\begin{cases} x, & 0\leqslant x<\frac{1}{2} \\ \frac{1}{2}, & x=\frac{1}{2} \\ 1-x, & \frac{1}{2}<x\leqslant 1\end{cases}$ at $x=\dfrac{1}{2}$

- LEFT HAND LIMIT *(x < 1/2)*

$$=\lim_{x\to\frac{1}{2}^-} f(x)$$

$$=\lim_{x\to\frac{1}{2}^-} x\left\{\text{put } x=\frac{1}{2}-h\right\}$$

$$=\lim_{h\to 0}\frac{1}{2}-h$$

$$=\frac{1}{2}-0$$

$$=\frac{1}{2}$$

- RIGHT HAND LIMIT *(x >1/2)*

$$=\lim_{x\to\frac{1}{2}^+} f(x)$$

$$=\lim_{x\to\frac{1}{2}^+} 1-x\left\{\text{put } x=\frac{1}{2}+h\right\}$$

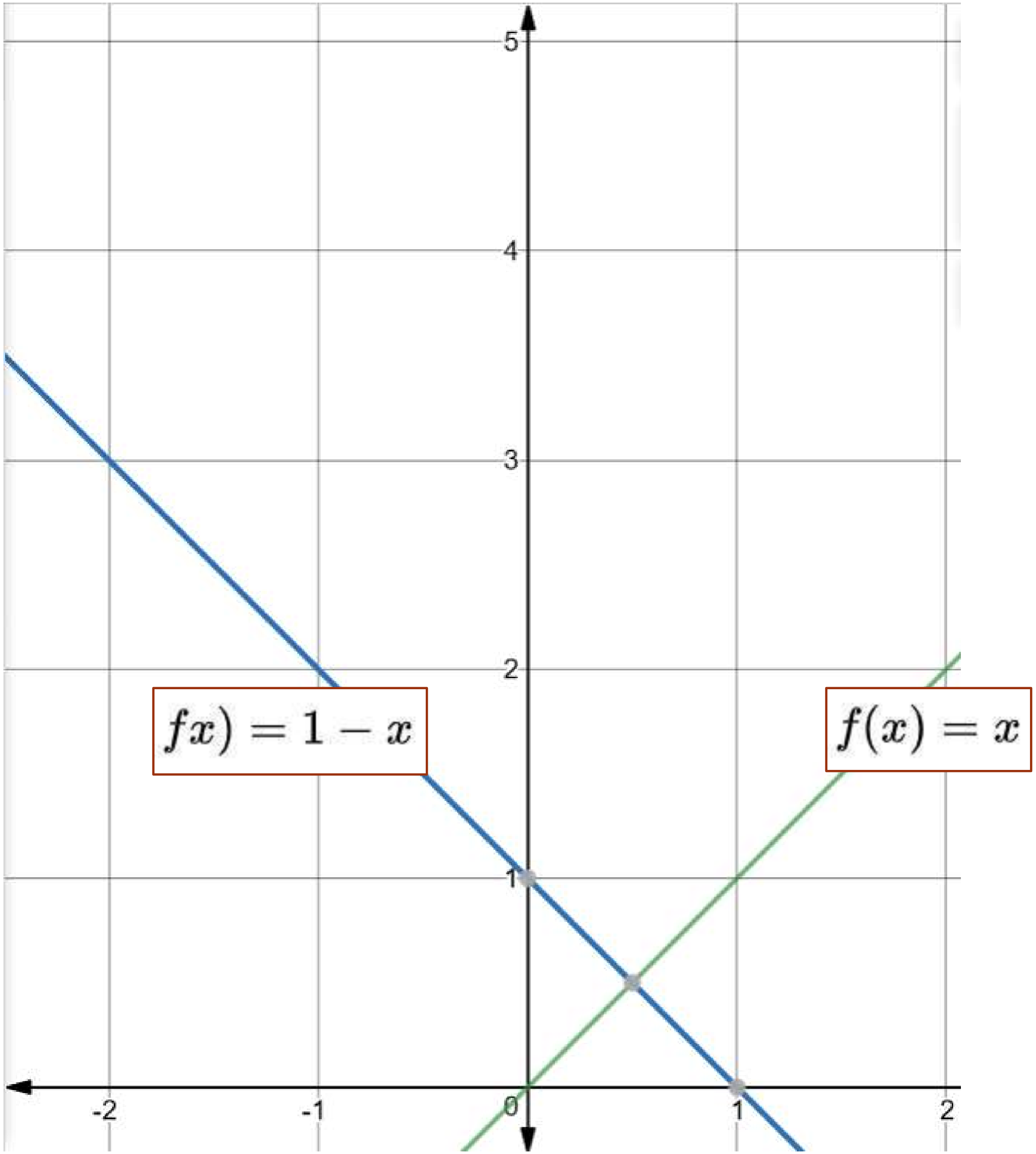
$$=\lim_{h\to 0} 1-\left(\frac{1}{2}+h\right)$$

$$=1-\left(\frac{1}{2}+0\right)$$

$$=\frac{1}{2}$$

L.H.L = R.H.L → $\lim_{x\to\frac{1}{2}} f(x)=\dfrac{1}{2}$ exists

Video links



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Q5

Show that $\lim_{x\to 0} \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$ doesn't exist

- LEFT HAND LIMIT

$$= \lim_{x\to 0^-} f(x)$$

$$= \lim_{x\to 0^-} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} \{\text{put } x = 0-h\}$$

$$= \lim_{h\to 0} \frac{e^{\frac{1}{(0-h)}}-1}{e^{\frac{1}{(0-h)}}+1} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{h\to 0} \frac{e^{\frac{-1}{h}}-1}{e^{\frac{-1}{h}}+1}$$

$$= \lim_{h\to 0} \frac{\frac{1}{e^{\frac{1}{h}}}-1}{\frac{1}{e^{\frac{1}{h}}}+1}$$

$$= \frac{0-1}{0+1}$$

$$= -1$$

- RIGHT HAND LIMIT

$$= \lim_{x\to 0^+} f(x)$$

$$= \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} \{\text{put } x = 0+h\}$$

$$= \lim_{h\to 0} \frac{e^{\frac{1}{(0+h)}}-1}{e^{\frac{1}{(0+h)}}+1} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{h\to 0} \frac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1}$$

$$= \lim_{h\to 0} \frac{1-\frac{1}{e^{\frac{1}{h}}}}{1+\frac{1}{e^{\frac{1}{h}}}}$$

$$= \frac{1-0}{1+0}$$

$$= 1$$

$h \to 0$

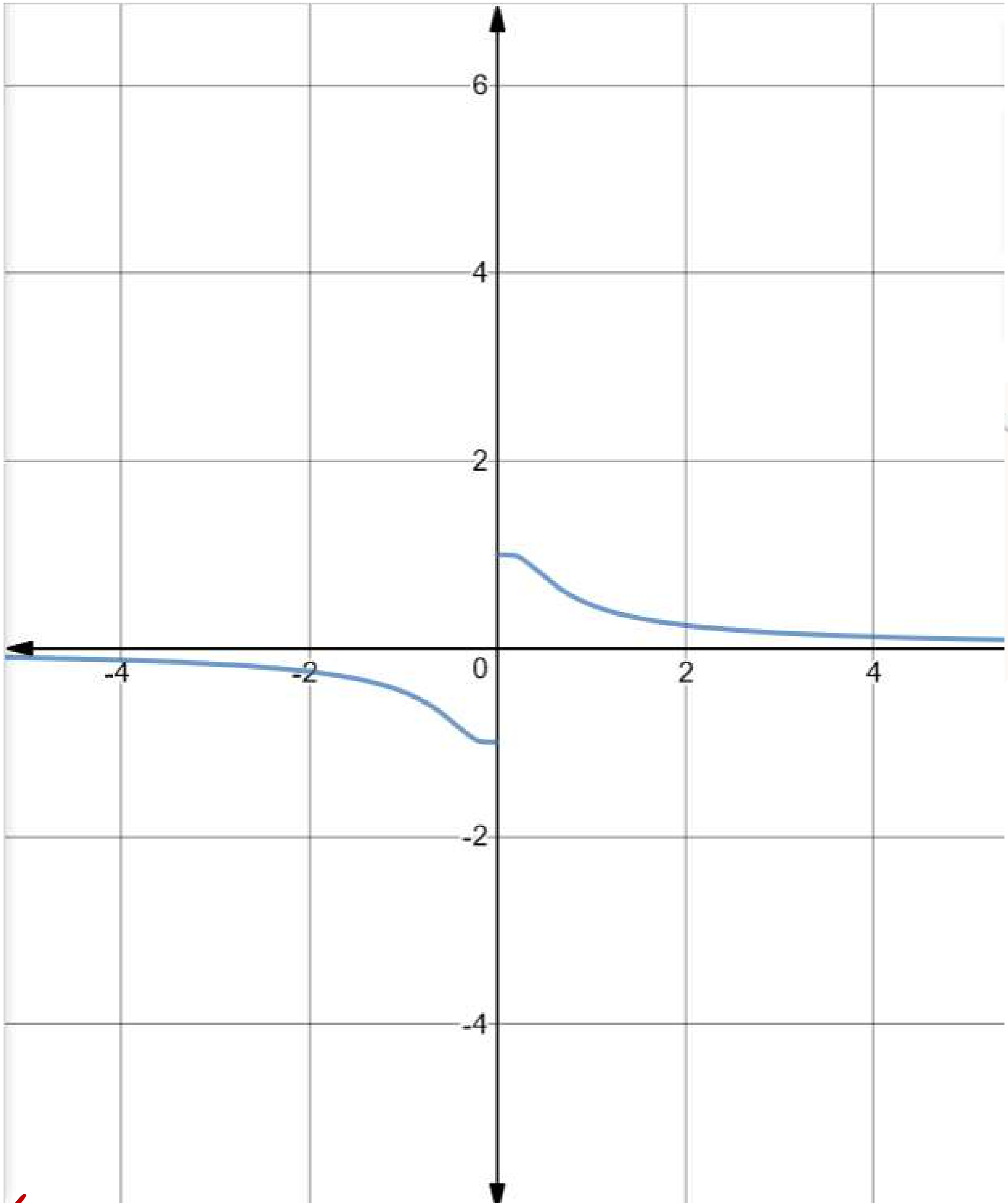
$\frac{1}{h} \to \infty$

$e^{\frac{1}{h}} \to \infty$

$\frac{1}{e^{\frac{1}{h}}} \to 0$

L.H.L ≠ R.H.L → $\lim_{x\to 0} f(x)$ doesn't exists

Video links



$$f(x) = \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$$

Video links

Greatest Integer function

$$[x] = \begin{cases} n, & x = n \\ n-1, & n-1 \leqslant x < n \end{cases}$$

[x] is known as greatest integer function

Example :

$[5] = 5$

$[3] = 3$

$[-3] = -3$

$\left[\dfrac{25}{3}\right] = [8.3] = 8 \quad \text{as} \;\; 8 < 8.3 < 9$

Video links

Q6 Examine the existence of $\lim_{x\to 3}[x]$

- LEFT HAND LIMIT

$$= \lim_{x\to 3^-}[x] \{\text{put } x = 3-h\}$$
$$= \lim_{h\to 0}[3-h]$$
$$= 2$$

$3-h = 2.9999$ (approximate)
$2 < 2.9999 < 3$
$2 < 3-h < 3$
so $[3-h] = 2$

- RIGHT HAND LIMIT

$$= \lim_{x\to 3^+}[x] \{\text{put } x = 3+h\}$$
$$= \lim_{h\to 0}[3+h]$$
$$= 3$$

$3+h \approx 3.0001$
$3 < 3.0001 < 4$
$3 < 3+h < 4$
so $[3+h] = 3$

L.H.L ≠ R.H.L → $\lim_{x\to 3}[x]$ doesn't exist

Video links

Q7 Examine the existence of $\lim_{x \to \frac{5}{2}} [x]$

- LEFT HAND LIMIT

$$= \lim_{x \to \left(\frac{5}{2}\right)^-} [x] \quad \left\{\text{put } x = \frac{5}{2} - h\right\}$$

$$= \lim_{h \to 0} \left[\frac{5}{2} - h\right]$$

$$= \lim_{h \to 0} [2.5 - h]$$

$$= 2$$

$$\frac{5}{2} - h \approx 2.4999$$
$$2 < 2.4999 < 3$$
$$2 < 2.5 - h < 3$$
$$\text{so } [2.5 - h] = 2$$

- RIGHT HAND LIMIT

$$= \lim_{x \to \left(\frac{5}{2}\right)^+} [x] \quad \left\{\text{put } x = \frac{5}{2} + h\right\}$$

$$= \lim_{h \to 0} \left[\frac{5}{2} + h\right]$$

$$= \lim_{h \to 0} [2.5 + h]$$

$$= 2$$

$$\frac{5}{2} + h \approx 2.5999$$
$$2 < 2.5999 < 3$$
$$2 < 2.5 + h < 3$$
$$\text{so } [2.5 + h] = 2$$

L.H.L = R.H.L → $\lim_{x \to \frac{5}{2}} [x]$ exists

Video links

Evaluation of Limit

Evaluation of limit is divided into two parts:

- Evaluation of algebraic limit.

  5 different methods

  1. Direct Substitution method
  2. Factorisation method
  3. Rationalisation method
  4. Evaluation of limit at infinity
  5. Evaluation of limit using some standard formulas.

- Evaluation of non-algebraic limit.

  1. **Evaluation of limit using some standard formulas.**

Video links

EVALUATION OF ALGEBRAIC LIMITS

5 different methods

1. Direct Substitution method
2. Factorisation method
3. Rationalisation method
4. Evaluation of limit at infinity
5. Evaluation of limit using some standard formulas.

Video links

1. Direct substitution method

Q1 Evaluate $\lim_{x \to 2} 4x^2 + 3$

Solution:

$$\lim_{x \to 2} 4x^2 + 3$$
$$= 4(2)^2 + 3$$
$$= 4(4) + 3$$
$$= 16 + 3$$
$$= 19$$

Q2 Evaluate $\lim_{x \to 2} \frac{x^2 + 3}{x - 1}$

Solution:

$$\lim_{x \to 2} \frac{x^2 + 3}{x - 1}$$
$$= \frac{(2)^2 + 3}{2 - 1}$$
$$= \frac{4 + 3}{1}$$
$$= 7$$

Video links

1. Direct substitution method

Q3 Evaluate $\lim_{x\to 0} \frac{\sqrt{1+x}+\sqrt{1-x}}{1-x}$

Solution: $\lim_{x\to 0} \frac{\sqrt{1+x}+\sqrt{1-x}}{1-x}$

$$= \frac{\sqrt{1+0}+\sqrt{1-0}}{1-0}$$

$$= \frac{\sqrt{1}+\sqrt{1}}{1}$$

$$= \frac{1+1}{1}$$

$$= \frac{2}{1}$$

$$= 2$$

Video links

2. Factorisation method

Q1 Evaluate $\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$

Solution:

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 4} \frac{(x - 4)(x + 4)}{x - 4}$$

$$= \lim_{x \to 4} x + 4$$

$$= 4 + 4$$

$$= 8$$

NOTE

If after substituting $x = a$ in $\lim_{x \to a} \frac{f(x)}{g(x)}$ gives $\frac{0}{0}$, then use factorisation methods.

Step1 $\rightarrow$ factorise either $f(x)$ or $g(x)$ or both.

Step2 $\rightarrow$ cancel out common factor if any.

Step3 $\rightarrow$ use direct substitution method again.

Video links

2. Factorisation method

Q2 Evaluate $\lim_{x \to 1} \frac{x^2 - 4x + 3}{x^2 - 6x + 5}$

Solution: $\lim_{x \to 1} \frac{x^2 - 4x + 3}{x^2 - 6x + 5} \quad \left(\frac{0}{0}\right)$

$$= \lim_{x \to 1} \frac{x^2 - 3x - x + 3}{x^2 - 5x - x + 5}$$

$$= \lim_{x \to 1} \frac{x(x-1) - 3(x-1)}{x(x-1) - 5(x-1)}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x-3)}{\cancel{(x-1)}(x-5)}$$

$$= \lim_{x \to 1} \frac{(x-3)}{(x-5)}$$

$$= \frac{1-3}{1-5} = \frac{-2}{-4} = \frac{1}{2}$$

Video links

3. Rationalisation method

NOTE

if there is a square root term either in Numerator and Denominator or both and after putting $x = a$ directly in $\lim_{x \to a} \frac{f(x)}{g(x)}$ gives $\frac{0}{0}$ form then use Rationalisation method.

METHOD

1. Multiply the conjugate of the square root term both in numerator and denominator.
2. Then simplify.

Video links

3. Rationalisation method

Q1 Evaluate $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$

Solution:

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}\quad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)}{x\left(\sqrt{x+1}+1\right)}$$

$$=\lim_{x\to 0}\frac{x+1-1}{x\left(\sqrt{x+1}+1\right)}$$

$$=\lim_{x\to 0}\frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1}+1\right)}$$

$$=\lim_{x\to 0}\frac{1}{\sqrt{x+1}+1}=\frac{1}{\sqrt{0+1}+1}=\frac{1}{\sqrt{1}+1}=\frac{1}{1+1}=\frac{1}{2}$$

Video links

3. Rationalisation method

Q2 Evaluate $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{2x}$

Solution:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{2x}\quad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}{2x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$=\lim_{x\to 0}\frac{\left(\sqrt{1+x}\right)^2-\left(\sqrt{1-x}\right)^2}{2x\left(\sqrt{1+x}+\sqrt{1-x}\right)}=\lim_{x\to 0}\frac{(1+x)-(1-x)}{2x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$=\lim_{x\to 0}\frac{1+x-1+x}{2x\left(\sqrt{1+x}+\sqrt{1-x}\right)}=\lim_{x\to 0}\frac{2x}{2x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$=\lim_{x\to 0}\frac{1}{\sqrt{1+x}+\sqrt{1-x}}=\frac{1}{\sqrt{1+0}+\sqrt{1-0}}=\frac{1}{\sqrt{1}+\sqrt{1}}=\frac{1}{2}$$

Video links

4. Evaluation of limit at infinity

METHOD

Step1 $\rightarrow$ the expression should be a rational function, if not convert it into a rational function i.e. $\dfrac{f(x)}{g(x)}$

Step2 $\rightarrow$ if k is the heighest power of x then divide each term of numerator & denominator by x^k.

Step3 $\rightarrow$ use $\lim_{x\to\infty} \dfrac{1}{x^k} = 0,\ k > 0.$

Video links

4. Evaluation of limit at infinity

Q1 Evaluate $\lim_{x\to\infty} \frac{3x^2+4x-1}{2x^2+x+2}$

Solution: $\lim_{x\to\infty} \frac{3x^2+4x-1}{2x^2+x+2}$

$$= \lim_{x\to\infty} \frac{\frac{3x^2}{x^2}+\frac{4x}{x^2}-\frac{1}{x^2}}{\frac{2x^2}{x^2}+\frac{x}{x^2}+\frac{2}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3+\frac{4}{x}-\frac{1}{x^2}}{2+\frac{1}{x}+\frac{2}{x^2}}$$

$$= \frac{\lim_{x\to\infty} 3+\lim_{x\to\infty}\frac{4}{x}-\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 2+\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}\frac{2}{x^2}}$$

$$= \frac{3+0-0}{2+0+0} = \frac{3}{2}$$

Video links

4. Evaluation of limit at infinity

Q2 Evaluate $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}-1}$

Solution: $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}-1}$

$$= \lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{\sqrt{x^2+1}}{x} - \frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{\frac{x^2+1}{x^2}} - \frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}} - \frac{1}{x}}$$

$$= \frac{1}{\sqrt{1+0}-0} = \frac{1}{\sqrt{1}} = \frac{1}{1} = 1$$

Video links

4. Evaluation of limit at infinity

Q3 Evaluate $\lim_{n\to\infty} \frac{1+2+3+\cdots+n}{n^2}$

Solution: $$\lim_{n\to\infty} \frac{1+2+3+\cdots+n}{n^2}$$

$$= \lim_{n\to\infty} \frac{\frac{n(n+1)}{2}}{n^2}$$

$$= \lim_{n\to\infty} \frac{n(n+1)}{2n^2}$$

$$= \lim_{n\to\infty} \frac{n^2+n}{2n^2}$$

$$= \lim_{n\to\infty} \frac{\frac{n^2}{n^2}+\frac{n}{n^2}}{\frac{2n^2}{n^2}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2} = \frac{1+0}{2} = \frac{1}{2}$$

Video links

4. Evaluation of limit at infinity

Q4 Evaluate

$$\lim_{n\to\infty} \frac{n!}{(n+1)! - n!}$$

Solution:

$$\lim_{n\to\infty} \frac{n!}{(n+1)! - n!}$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)n! - n!}$$

$$= \lim_{n\to\infty} \frac{n!}{n!(n+1-1)}$$

$$= \lim_{n\to\infty} \frac{1}{1(n+1-1)}$$

$$= \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

Video links

5. Evaluation of limit using standard formulas

FORMULAS

1. $\lim\limits_{x\to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}, a > 0$

2. $\lim\limits_{x\to 0} (1 + x)^{\frac{1}{x}} = e$

3. $\lim\limits_{x\to \infty} \left(1 + \dfrac{1}{x}\right)^{x} = e$

4. $\lim\limits_{x\to \infty} (1 + \lambda x)^{\frac{1}{x}} = e^{\lambda}$

5. $\lim\limits_{x\to \infty} \left(1 + \dfrac{\lambda}{x}\right)^{x} = e^{\lambda}$

Video links

5. Evaluation of limit using standard formulas

Q1 Evaluate $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

Solution: $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

$= \lim_{x \to 3} \frac{(x)^2 - (3)^2}{x - 3}$

$= 2(3)^{2-1}$

$= 2(3)$

$= 6$

FORMULA

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}, a > 0$$

$n = 2$

$a = 3$

Video links

5. Evaluation of limit using standard formulas

Q2 Evaluate $\lim_{x \to 0} \frac{(x+9)^{\frac{3}{2}} - 27}{x}$

Solution: $\lim_{x \to 0} \frac{(x+9)^{\frac{3}{2}} - 27}{x}$

$= \lim_{x \to 0} \frac{(x+9)^{\frac{3}{2}} - (9)^{\frac{3}{2}}}{(x+9) - 9}$

variable is $x + 9$

$x \to 0$

$x + 9 \to 9$

$= \lim_{x+9 \to 9} \frac{(x+9)^{\frac{3}{2}} - (9)^{\frac{3}{2}}}{(x+9) - 9}$

FORMULA

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}, a > 0$$

$n = \frac{3}{2}$

$a = 9$

$= \frac{3}{2}(9)^{\frac{3}{2}-1}$

$= \frac{3}{2}(9)^{\frac{1}{2}}$

$= \frac{3}{2}(3) = \frac{9}{2}$

Video links

EVALUATION OF NON-ALGEBRAIC LIMITS

FORMULAS

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
2. $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$
3. $\lim_{x \to 0} \frac{\tan x}{x} = 1$
4. $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$
5. $\lim_{x \to 0} \frac{\log(x+1)}{x} = 1$
6. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
7. $\lim_{x \to 0} \frac{a^x - 1}{x} = \log a \quad (a > 0)$

Video links

Evaluation of non-algebraic limits

Q1 Evaluate $\lim_{x \to 0} \frac{\sin 4x}{x}$

Solution:

$$\lim_{x \to 0} \frac{\sin 4x}{x}$$

$$= \lim_{x \to 0} \frac{4 \sin 4x}{4x}$$

$$= 4 \lim_{4x \to 0} \frac{\sin 4x}{4x}$$

$$= 4(1)$$

$$= 4$$

$x \to 0$

$4x \to 0$

FORMULA

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Video links

Evaluation of non-algebraic limits

Q2 Evaluate $\lim_{x \to 0} \frac{\sin 5x}{\tan 3x}$

Solution:

$$\lim_{x \to 0} \frac{\sin 5x}{\tan 3x}$$

$$= \frac{\lim_{x \to 0} \sin 5x}{\lim_{x \to 0} \tan 3x}$$

$$= \frac{\lim_{x \to 0} \frac{\sin 5x}{5x} * \frac{5x}{1}}{\lim_{x \to 0} \frac{\tan 3x}{3x} * \frac{3x}{1}}$$

$$= \frac{5}{3} \frac{\lim_{5x \to 0} \frac{\sin 5x}{5x}}{\lim_{3x \to 0} \frac{\tan 3x}{3x}}$$

$$= \frac{5}{3} * \frac{1}{1} = \frac{5}{3}$$

$$x \to 0$$
$$5x \to 0$$
$$3x \to 0$$

FORMULA

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{\tan x}{x} = 1$$

Video links

Evaluation of non-algebraic limits

Q3 Evaluate $$\lim_{x \to 0} \frac{1+\cos x}{x^2+1}$$

Solution: $$\lim_{x \to 0} \frac{1+\cos x}{x^2+1}$$

$$= \frac{1+\cos 0}{0^2+1}$$

$$= \frac{1+1}{1}$$

$$= \frac{2}{1} = 2$$

Video links

Evaluation of non-algebraic limits

Q4 Evaluate $\lim_{x\to 0} \frac{e^{\sin x}-1}{x}$

Solution: $\lim_{x\to 0} \frac{e^{\sin x}-1}{x}$

$$= \lim_{x\to 0} \frac{e^{\sin x}-1}{x * \sin x} * \frac{\sin x}{1}$$

$$= \lim_{x\to 0} \frac{e^{\sin x}-1}{\sin x} * \lim_{x\to 0} \frac{\sin x}{x}$$

$$= \lim_{\sin x\to 0} \frac{e^{\sin x}-1}{\sin x} * 1$$

$$= 1 * 1 = 1$$

$x \to 0$

$\sin x \to 0$

FORMULA

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{e^{x}-1}{x} = 1$$

Video links

Evaluation of non-algebraic limits

Q5 Evaluate $\lim_{x\to 0} \frac{\csc x - \cot x}{x}$

Solution: $\lim_{x\to 0} \frac{\csc x - \cot x}{x}$

$$= \lim_{x\to 0} \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x}$$

$$= \lim_{x\to 0} \frac{1 - \cos x}{x * \sin x}$$

$$= \lim_{x\to 0} \frac{(1 - \cos x)}{x * \sin x} * \frac{\sin x}{\sin x}$$

$$= \lim_{x\to 0} \frac{(1 - \cos x)\sin x}{x * \sin^2 x}$$

$$= \lim_{x\to 0} \frac{(1 - \cos x)\sin x}{x * \sin^2 x}$$

$$= \lim_{x\to 0} \frac{(1 - \cos x)\sin x}{x(1 - \cos^2 x)}$$

$$= \lim_{x\to 0} \frac{(1 - \cos x)\sin x}{x(1 - \cos x)(1 + \cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} * \frac{1}{(1 + \cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} * \lim_{x\to 0} \frac{1}{(1 + \cos x)}$$

$$= 1 * \frac{1}{1 + \cos 0} = \frac{1}{1 + 1} = \frac{1}{2}$$

FORMULA

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Video links

Evaluation of non-algebraic limits

Q6 Evaluate $\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}$

Solution:

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}$$

$$=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{\sin^3 x}$$

$$=\lim_{x\to 0}\frac{\sin x-\sin x*\cos x}{\sin^3 x*\cos x}$$

$$=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{\sin^3 x*\cos x}$$

$$=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{\sin^3 x*\cos x}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x*\cos x}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x*\cos x}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x*\cos x}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{(1-\cos^2 x)\cos x}$$

$$=\lim_{x\to 0}\frac{\cancel{1-\cos x}}{\cancel{(1-\cos x)}(1+\cos x)\cos x}$$

$$=\lim_{x\to 0}\frac{1}{(1+\cos x)\cos x}$$

$$=\frac{1}{(1+\cos 0)\cos 0}=\frac{1}{(1+1)1}=\frac{1}{2}$$

Video links

Evaluation of non-algebraic limits

Q7 Evaluate $\lim_{x \to 1} \frac{\log(2x-1)}{x-1}$

Solution: $\lim_{x \to 1} \frac{\log(2x-1)}{x-1}$

$= \lim_{x \to 1} \frac{\log(2x-2+1)}{x-1}$

$= \lim_{x \to 1} \frac{\log\{2(x-1)+1\}}{x-1}$

$= \lim_{x \to 1} \frac{2}{2} * \frac{\log\{2(x-1)+1\}}{x-1}$

$= 2 \lim_{x \to 1} \frac{\log\{2(x-1)+1\}}{2(x-1)}$

$= 2 \lim_{2(x-1) \to 0} \frac{\log\{2(x-1)+1\}}{2(x-1)} = 2 * 1 = 2$

$x \to 1$
$x - 1 \to 0$
$2(x-1) \to 0$

FORMULA

$\lim_{x \to 0} \frac{\log(x+1)}{x} = 1$

Video links

Evaluation of non-algebraic limits

Q8 Evaluate $$\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}$$

Solution:

$$\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}$$

$$=\lim_{h\to 0}\frac{\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}}{h}$$

$$=\lim_{h\to 0}\frac{\sin(x+h)*\cos x-\sin x*\cos(x+h)}{\cos(x+h)*\cos x*h}$$

$$=\lim_{h\to 0}\frac{\sin(x+h-x)}{\cos(x+h)*\cos x*h}=\lim_{h\to 0}\frac{\sin h}{h}*\frac{1}{\cos(x+h)*\cos x}$$

$$=\lim_{h\to 0}\frac{\sin h}{h}*\lim_{h\to 0}\frac{1}{\cos(x+h)*\cos x}=1*\frac{1}{\cos(x+0)*\cos x}$$

$$=\frac{1}{\cos x*\cos x}=\frac{1}{\cos^2 x}=\sec^2 x$$

FORMULA

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Video links

-: CONTINUITY AND DISCONTINUITY OF FUNCTIONS :-

Definition :-

A function f(x) is said to be continuous at $x = a$ if

(i) limiting value exists. (i.e. L.H.L = R.H.L)

$\Rightarrow \lim_{x \to a} f(x)$ exists.

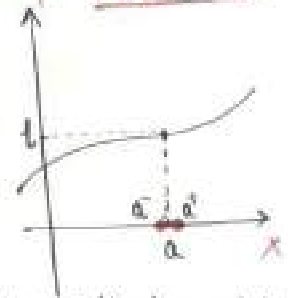
(ii) $f(a)$ exists (i.e. functional value exists)

(iii) $\lim_{x \to a} f(x) = f(a)$.

NOTE :- In case of at least 'one' of the above condition fails, then the function is discontinuous at $x = a$.

Geometrical Representation :-

① continuous

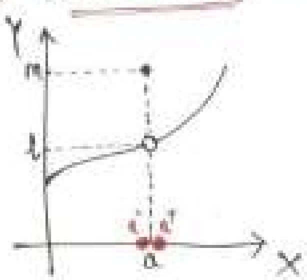


Hence (i) $\lim_{x \to a} f(x)$ exists.

(ii) $f(a)$ exists

(iii) Both are equal.

② Discontinuous

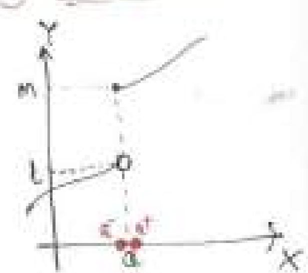


Here $\lim_{x \to a} f(x) = l$

(ii) $f(a) = m$

(iii) But both are not equal.

③ Discontinuous



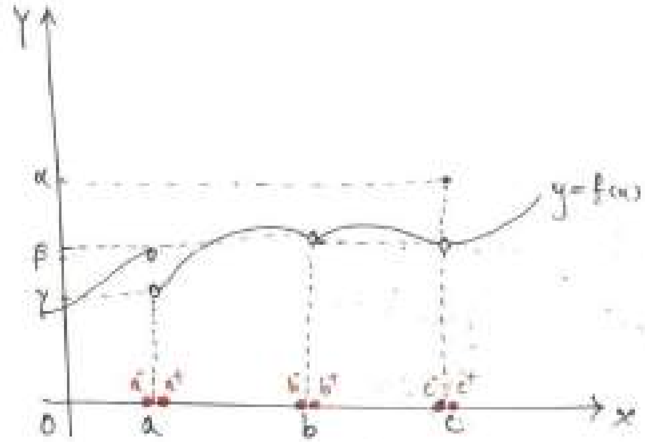
Here (i) $\lim_{x \to a^-} = l$ &

$\lim_{x \to a^+} = m$

$\Rightarrow$ limit doesn't exist

(ii) $f(a) = m$

Reasons of Discontinuity



At all the above three pts (i.e. a, b, c) function $y = f(x)$ is dis-continuous.

(1) At $x = a$:- In this pt. L.H.L ≠ R.H.L

as $\lim_{x \to a^-} = \beta$ and $\lim_{x \to a^+} = \gamma$

$\Rightarrow$ limit does not exist, so f(x) is discontinuous at $x = a$.

(2) At $x = b$:- At pt. 'b' L.H.L = R.H.L

as $\lim_{x \to b^-} = \beta$ and $\lim_{x \to b^+} = \beta \Rightarrow$ limit exists.

But $f(b)$ is not defined.

So f(x) is discontinuous at $x = b$.

(3) At $x = c$:- At pt. 'c' (i) L.H.L = R.H.L. as $\lim_{x \to c^-} = \beta = \lim_{x \to c^+}$

(ii) $f(c)$ is defined i.e. $f(c) = \alpha$

But $\lim_{x \to c} f(x) \neq f(c)$

So f(x) is discontinuous at $x = c$.

* Examine the continuity of each of the followings:

Q.1 $f(x) = \begin{cases} x^2+2, & x>1 \\ 2x+1, & x=1 \\ 3, & x<1 \end{cases}$ at $x=1$

Solution:-

Case I :- (limiting value).

$(x<1)$ L.H.L (at $x=1$)

$\lim_{x\to 1^-} f(x)$

$= \lim_{x\to 1^-} 3$

put $x=1-h$

$= \lim_{h\to 0} 3$

$= 3$

R.H.L (at $x=1$) $(x>1)$

$\lim_{x\to 1^+} f(x)$

$= \lim_{x\to 1^+} x^2+2$

put $x=1+h$

$= \lim_{h\to 0} (1+h)^2+2$

$= (1+0)^2+2 = 3$

as L.H.L = R.H.L

$\Rightarrow \lim_{x\to 1} f(x)$ exists.

and $\lim_{x\to 1} f(x) = 3$.

Case II :- (functional value).

At $x=1$, $f(x) = 2x+1$

$\Rightarrow f(1) = 2(1)+1$

$= 3$

Case III :- $\lim_{x\to 1} f(x) = 3 = f(1)$

$\therefore f(x)$ is continuous at $x=1$.

Q.2 $f(x) = \begin{cases} x - \frac{|x|}{x}, & x \neq 0 \\ 2, & x=0 \end{cases}$ at $x=0$

Solution:-

Case I :- (limiting value).

L.H.L (at $x=0$)

$\lim_{x\to 0^-} f(x)$

$= \lim_{x\to 0^-} x - \frac{|x|}{x}$

put $x = 0-h = -h$

$= \lim_{h\to 0} -h - \frac{|-h|}{-h}$

$= \lim_{h\to 0} -h - \frac{h}{-h}$

$= \lim_{h\to 0} -h+1$

$= 0+1$

$= 1$

R.H.L (at $x=0$)

$\lim_{x\to 0^+} f(x)$

$= \lim_{x\to 0^+} x - \frac{|x|}{x}$

put $x = 0+h = h$

$= \lim_{h\to 0} h - \frac{|h|}{h}$

$= \lim_{h\to 0} h-1$

$= 0-1$

$= -1$

Here L.H.L $\neq$ R.H.L

$\Rightarrow \lim_{x\to 0} f(x)$ doesn't exists

So $f(x)$ is dis continuous at $x=0$.

Q.3 $f(x) = \begin{cases} \frac{x^2-9}{x-3} & , x \neq 3 \\ 6 & , x = 3 \end{cases}$ at $x = 3$

Solution :-

Case I (Limiting Value)

$\lim_{x \to 3} f(x)$

$= \lim_{x \to 3} \frac{x^2-9}{x-3} \left(\frac{0}{0}\right)$

$= \lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)}$

$= \lim_{x \to 3} x+3$

$= 3+3$

$= 6$

$\Rightarrow \lim_{x \to 3} f(x) = 6$

Case II (functional value)

At $x = 3, f(x) = 6$

$\Rightarrow f(3) = 6$

Case III $\lim_{x \to 3} f(x) = 6 = f(3)$

$\therefore$ So $f(x)$ is continuous at $x = 3$.

Q.4 For what value of K the function

$f(x) = \begin{cases} \frac{\sin 2x}{x} & , x \neq 0 \\ K & , x = 0 \end{cases}$ at $x = 0$

Solution :-

Case I (Limiting Value)

$\lim_{x \to 0} f(x)$

$= \lim_{x \to 0} \frac{\sin 2x}{x}$

$= \lim_{x \to 0} 2 \cdot \frac{\sin 2x}{2x}$

$= 2 \lim_{2x \to 0} \frac{\sin 2x}{2x}$ (as $x \to 0$, $2x \to 0$)

$= 2 \times 1$

$= 2$

$\Rightarrow \lim_{x \to 0} f(x) = 2$

Case II (functional value)

at $x = 0$ $f(x) = K$

$\Rightarrow f(0) = K$

Case III It is given that $f(x)$ is continuous at $x = 0$

$\Rightarrow \lim_{x \to 0} f(x) = f(0)$

$\Rightarrow \boxed{2 = K}$ (Ans)

Q.5 For what value of 'a' and 'b'

$$f(x) = \begin{cases} ax^2+b, & x<1 \\ 1, & x=1 \\ 2ax-b, & x>1 \end{cases}$$ is continuous at $x=1$.

Solution:-

Case I (limiting value)

L.H.L (at $x=1$) ($x<1$)

$\lim_{x\to1^-} f(x)$

$= \lim_{x\to1^-} ax^2+b$

put $x=1-h$

$= \lim_{h\to0} a(1-h)^2+b$

$= a(1-0)^2+b$

$= a+b$

R.H.L (at $x=1$) ($x>1$)

$\lim_{x\to1^+} f(x)$

$= \lim_{x\to1^+} 2ax-b$

put $x=1+h$

$= \lim_{h\to0} 2a(1+h)-b$

$= 2a(1+0)-b$

$= 2a-b$

It is given that $f(x)$ is continuous at $x=1$

$\Rightarrow \lim_{x\to1} f(x)$ exists

or L.H.L = R.H.L

$\Rightarrow a+b = 2a-b.$

Case II (functional value)

at $x=1$ $f(x)=1$

$\Rightarrow f(1)=1$

Case III $\lim_{x\to1} f(x) = f(1)$ (as $f(x)$ is continuous)

$\Rightarrow \lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = f(1)$

$\Rightarrow a+b = 2a-b = 1$

from above $a+b=1$ —— (I)

$2a-b=1$ —— (II)

Solving (I) & (II) $\Rightarrow 3a=2$

$\boxed{a = 2/3}$

putting $a=2/3$ in eqn (I)

$2/3+b=1$

$\Rightarrow b = 1-2/3 = \frac{3-2}{3}$

$\Rightarrow \boxed{b=\frac{1}{3}}$

Q.6 Show that $f(x) = \begin{cases} x\sin\frac{1}{x}, & x\neq0 \\ 0, & x=0 \end{cases}$ is continuous at $x=0$.

Solution:-

Case I (limiting value)

$\lim_{x\to0} f(x)$

$= \lim_{x\to0} x\sin\frac{1}{x}$

$= \lim_{x\to0} x \times \lim_{x\to0} \sin\frac{1}{x}$

$= 0 \times$ a finite quantity

$= 0$

Thus, $\lim_{x \to 0} f(x) = 0$

Case II (functional value)

At $x=0$, $f(x) = 0$

$\Rightarrow f(0) = 0$

Case III finally, $\lim_{x \to 0} f(x) = 0 = f(0)$

Hence, $f(x)$ is continuous at $x=0$

Q.7 :- Examine the continuity of the function:

$$f(x) = \begin{cases} (1+2x)^{1/x}, & x \neq 0 \\ e^2, & x = 0 \end{cases} \text{ at } x=0$$

Solution :-

Case I :- (limiting value)

$\lim_{x \to 0} f(x)$

$= \lim_{x \to 0} (1+2x)^{1/x}$

$= \lim_{x \to 0} \left\{ (1+2x)^{\frac{1}{2x}} \right\}^2$

as $x \to 0 \Rightarrow 2x \to 0$

$= \lim_{2x \to 0} \left\{ (1+2x)^{1/2x} \right\}^2$

$= e^2$ (using $\lim_{x \to 0} (1+x)^{1/x} = e$)

Case II :- (functional value)

at $x=0$ $f(x) = e^2$

$f(0) = e^2$

Case III :- $\lim_{x \to 0} f(x) = e^2 = f(0)$

Thus, $f(x)$ is continuous at $x=0$.

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Chapter-2

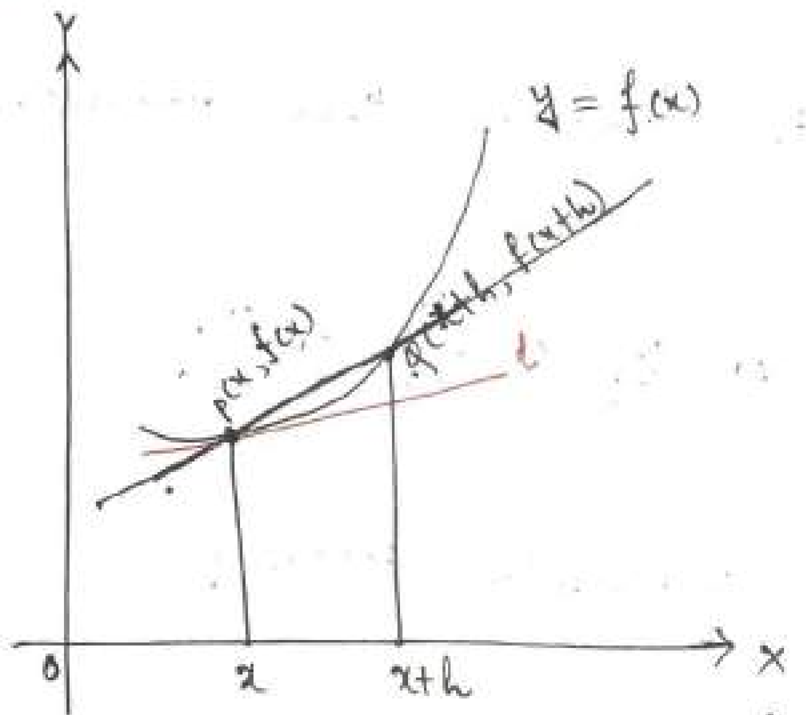
-: DERIVATIVES :-

Concept of Derivative: ① Derivative means the rate of change of a function with respect to a variable.

or

② Geometrically, Derivative means the slope of the tangent of the curve at a pt. 'p'.

Geometrical Interpretation of Derivative:-



Now slope of secant $PQ = \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{\text{change in } y}{\text{change in } x}$.

$$\Rightarrow \text{slope of } PQ = \frac{f(x+h)-f(x)}{h}$$

∴ Let's approach h towards 0

i.e. $h \to 0$

$\Rightarrow Q \to P$

Then the secant PQ becomes the line L which is the tangent to the curve $y = f(x)$

Then slope of the tangent

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

which is the derivative of the function at the pt. 'P'.

Notations of derivative :-

$y = f(x)$ be the function, then derivative is denoted by

y' or $f'(x)$ or y_1 or $\frac{dy}{dx}$ or Dy

* From the Geometrical Meaning we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

→ Known as first principle Method to find derivative

or known as Δ-Method.

Standard formulas of Derivative :-

	function y or $f(x)$	Derivative $\frac{dy}{dx}$ or $f'(x)$	
1	x^n	$n x^{n-1}$	Derivative of algebraic function.
2	x	1	
3	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	
4	$\frac{1}{x}$	$-\frac{1}{x^2}$	
5	K (constant)	0	
6	$\log_e x$	$\frac{1}{x}$	Derivative of Logarithmic function.
7	$\log_a x$	$\frac{1}{x \log a}$	
8	e^x	e^x	Derivative of exponential function.
9	a^x	$a^x \log a$	
10	$\sin x$	$\cos x$	Derivative of Trigonometric function.
11	$\cos x$	$-\sin x$	
12	$\tan x$	$\sec^2 x$	
13	$\cot x$	$-\csc^2 x$	
14	$\sec x$	$\sec x \cdot \tan x$	
15	$\csc x$	$-\csc x \cdot \cot x$	

	function y or $f(x)$	Derivative $\frac{dy}{dx}$ or $f'(x)$
16	$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$
17	$\cos^{-1}x$	$\frac{-1}{\sqrt{1-x^2}}$
18	$\tan^{-1}x$	$\frac{1}{1+x^2}$
19	$\cot^{-1}x$	$\frac{-1}{1+x^2}$
20	$\sec^{-1}x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
21	$\csc^{-1}x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$

} Derivative of Inverse trigonometric function

* Derivative of the above functions are actually obtained by using the first principle method of derivative.

i.e. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

For example :- ① $y = x^n$ then $f'(x) = ?$

Solution:- Given $f(x) = x^n$

$$f(x+h) = (x+h)^n$$

By first principle of derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^n - x^n}{(x+h)-x}$$

Now as $h \to 0$

$\Rightarrow x+h \to x$

$$= \lim_{x+h\to x} \frac{(x+h)^n - x^n}{x+h-x}$$

$$= nx^{n-1}$$

$\left[\text{by using } \lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$

② $y = \sin x$, find $f'(x) = ?$

Solution:- Given $f(x) = \sin x$.

$$f(x+h) = \sin(x+h).$$

By using first principle of derivative

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{x+h+x}{2}\right)\cdot \sin\left(\frac{x+h-x}{2}\right)}{h} \quad \left[\sin C - \sin D = 2\cos\frac{C+D}{2}\cdot \sin\frac{C-D}{2}\right]$$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\cdot \sin h/2}{h}$$

$$= \lim_{h\to 0} \frac{\cos\left(\frac{2x+h}{2}\right)\cdot \sin h/2}{h/2}$$

$$= \lim_{h\to 0} \cos\left(\frac{2x+h}{2}\right)\cdot \lim_{h\to 0}\frac{\sin h/2}{h/2}$$

$$= \cos\left(\frac{2x+0}{2}\right)\cdot 1 \quad \left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= \cos\left(\frac{2x}{2}\right)$$

$$= \cos x$$

So if $f(x) = \sin x$

$\Rightarrow f'(x) = \cos x$

③ If $f(x) = e^x$, then find $f'(x) = ?$

Solution:- Given $f(x) = e^x$

$f(x+h) = e^{x+h}$

using 1st principle of derivative

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x\cdot e^h - e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x(e^h-1)}{h}$$

$$= e^x \lim_{h\to 0}\frac{e^h-1}{h}$$

$$= e^x \times 1 \quad \left(\because \lim_{x\to 0}\frac{e^x-1}{x} = 1\right)$$

$$= e^x$$

for $f(x) = e^x$

$f'(x) = e^x$

Theorems of derivative:-

① $\frac{d}{dx}\{f(x)+g(x)\} = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$ [Addition Rule]

② $\frac{d}{dx}\{f(x)-g(x)\} = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$ [Substraction Rule]

③ $\frac{d}{dx}\{f(x)\cdot g(x)\} = \left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right]$

(product Rule)

④ $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\left[\frac{d}{dx}f(x)\right]g(x) - f(x)\left[\frac{d}{dx}g(x)\right]}{\left[g(x)\right]^2}$

(Quotient Rule)

⑤ $\frac{d}{dx}\{k\,f(x)\} = k\left\{\frac{d}{dx}f(x)\right\}$

Q:- Evaluate the derivative of the followings:

(i) $y = \sin x - x^3 + \log x$.

Sol^n:- $\frac{dy}{dx} = \frac{d}{dx}(\sin x - x^3 + \log x)$

$= \frac{d}{dx}\sin x - \frac{d}{dx}x^3 + \frac{d}{dx}\log x$

$= \cos x - 3x^2 + \frac{1}{x}$

(ii) $y = 3^x + \sin x - e^x$

Sol^n:- $\frac{dy}{dx} = \frac{d}{dx}(3^x + \sin x - e^x)$

$= \frac{d}{dx}3^x + \frac{d}{dx}\sin x - \frac{d}{dx}e^x$

$= 3^x \log 3 + \cos x - e^x$

(iii) $y = 9x^2 + \frac{3}{x} + 5\sec x$

Sol^n:- $\frac{dy}{dx} = \frac{d}{dx}\left(9x^2 + \frac{3}{x} + 5\sec x\right)$

$= \frac{d}{dx}(9x^2) + \frac{d}{dx}\frac{3}{x} + \frac{d}{dx}5\sec x$

$= 9\left(\frac{d}{dx}x^2\right) + 3\left(\frac{d}{dx}\frac{1}{x}\right) + 5\left(\frac{d}{dx}\sec x\right)$

$= 9(2x) + 3\left(-\frac{1}{x^2}\right) + 5(\sec x\cdot\tan x)$

$= 18x - \frac{3}{x^2} + 5\sec x\tan x$

(iv) $y = x^2\cos x$

Sol^n $\frac{dy}{dx} = \frac{d}{dx}\{x^2\cos x\}$

$= \left[\frac{d}{dx}x^2\right]\cos x + x^2\left[\frac{d}{dx}\cos x\right]$

$= 2x\cdot\cos x + x^2(-\sin x)$

$= 2x\cdot\cos x - x^2\sin x$.

(v) $y = \frac{a^x - b^x}{x}$

Sol^n:- $\frac{dy}{dx} = \frac{d}{dx}\left\{\frac{a^x - b^x}{x}\right\}$

$= \frac{\left[\frac{d}{dx}(a^x - b^x)\right] x - (a^x - b^x)\left[\frac{d}{dx} x\right]}{x^2}$

$= \frac{(a^x \log a - b^x \log b)\, x - (a^x - b^x)}{x^2}$

$= \frac{x a^x \log a - x\, b^x \log b - a^x + b^x}{x^2}$

$= \frac{a^x (x \log a - 1) + b^x (1 - x \log b)}{x^2}$

(vi) $y = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$

Sol^n:- $\frac{dy}{dx} = \frac{d}{dx}\left\{\frac{\sqrt{x} - 1}{\sqrt{x} + 1}\right\}$

$= \frac{\left[\frac{d}{dx}(\sqrt{x} - 1)\right](\sqrt{x} + 1) - (\sqrt{x} - 1)\left[\frac{d}{dx}(\sqrt{x} + 1)\right]}{\{\sqrt{x} + 1\}^2}$

$= \frac{\frac{1}{2\sqrt{x}}(\sqrt{x} + 1) - (\sqrt{x} - 1)\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{x} + 1)^2}$

$= \frac{\frac{1}{2\sqrt{x}}\{\sqrt{x} + 1 - \sqrt{x} + 1\}}{(\sqrt{x} + 1)^2}$

$= \frac{\frac{1}{2\sqrt{x}}(2)}{(\sqrt{x} + 1)^2}$

$= \frac{1}{\sqrt{x}(\sqrt{x} + 1)^2}$

(vii) $y = \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}$

Sol^n $\frac{dy}{dx} = \frac{d}{dx}\sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}$

$= \frac{d}{dx}\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}$

$= \frac{d}{dx}\sqrt{\tan^2 x}$

$= \frac{d}{dx}\tan x$

$= \sec^2 x$

Derivative of composite function:- (Not a standard function)

composite function means function of functions.

i.e. $y = f[g(h(x))]$

And to find derivative of composite function we use **chain Rule.**

Or we can say chain Rule is used if the function is not a standard function.

For example: ① $y = (x^2+5)^5$ then find $\frac{dy}{dx}$

Solution:- $y = (x^2+5)^5$ which is a composite func (or Not coming under 21 standard formulas)

so let $u = x^2+5$

Then $y = u^5$ (which is in standard form)

Diff. both sides w.r.t 'u'

$$\frac{dy}{du} = \frac{d}{du}u^5$$

$$= 5u^4 \text{ --- (I)}$$

Again $u = x^2+5$

Diff. both sides w.r.t 'x'

$$\frac{du}{dx} = \frac{d}{dx}(x^2+5)$$

$$= 2x \text{ --- (II)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5u^4 \times 2x$$

$$= 5(x^2+5)^4 \cdot 2x$$

② $y = \log(\sin x)$, find $\frac{dy}{dx}$

Given function is a composite function.

so let $\sin x = u$

Then $y = \log u$ (which is in standard form)

Diff. both sides w.r.t u

$$\frac{dy}{du} = \frac{d}{du}\log u$$

$$= \frac{1}{u} \text{ --- (I)}$$

Again $u = \sin x$

Diff. both sides w.r.t x

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}\sin x$$

$$= \cos x \text{ --- (II)}$$

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times \cos x$

$$= \frac{1}{\sin x} \cdot \cos x = \cot x.$$

③ $y = \sin(\tan(x^4))$, find $\frac{dy}{dx}$.

Solution:-

$y = \sin(\tan x^4)$ [a composite function]

let $u = \tan x^4$

$y = \sin u$ [which is in standard form]

$$\frac{dy}{du} = \frac{d}{du}\sin u$$

$$\frac{dy}{du} = \cos u \text{ --- (I)}$$

Again $u = \tan x^4$ (is composite func)

so let $v = x^4$

$\Rightarrow u = \tan v$ (is in standard form)

$$\Rightarrow \frac{du}{dv} = \frac{d}{dv}\tan v$$

$$\Rightarrow \frac{du}{dv} = \sec^2 v \text{ --- (II)}$$

Again $v = x^4$ is already in standard form.

$$\text{so } \frac{dv}{dx} = \frac{d}{dx}x^4$$

$$\Rightarrow \frac{dv}{dx} = 4x^3 \text{ --- (III)}$$

So finally.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$$

$$= \cos u \times \sec^2 v \times 4x^3$$

$$= \cos(\tan x^4) \cdot \sec^2 x^4 \cdot 4x^3 \text{ (Ans)}.$$

As $\frac{dy}{dx}$ is obtained as $\frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$ which forms a chain

that's why the Method is known as Chain Rule.

Short cut Method to find derivative of composite funct.

Shortcut-1: $\frac{dy}{dx} = \left(\text{Derivative of outside function}\right) \times \left(\text{Derivative of Inside function}\right)$

Q-1 If $y = (x^2+5x)^6$ find $\frac{dy}{dx} = ?$

Solution:- Given $y = (x^2+5x)^6$ (a composite function).

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2+5x)^6 = 6(x^2+5x)^5 \times \frac{d}{dx}(x^2+5x)$$

$$= 6(x^2+5x)^5 \times (2x+5).$$

Here the outer function is power 6 and inner function is (x^2+5x).

Q-2 $y = \sin(\tan\sqrt{x})$, find $\frac{dy}{dx} = ?$

Solution:- Given $y = \sin(\tan\sqrt{x}) \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\sin(\tan\sqrt{x})$

$$\Rightarrow \frac{dy}{dx} = \cos(\tan\sqrt{x}) \times \frac{d}{dx}\tan\sqrt{x}$$

$$= \cos(\tan\sqrt{x}) \times \sec^2\sqrt{x} \times \frac{d}{dx}\sqrt{x}$$

$$= \cos(\tan\sqrt{x}) \cdot \sec^2\sqrt{x} \cdot \frac{1}{2\sqrt{x}}$$

[Here outer most is sin then tan then $\sqrt{x}$.]

Q-3 $y = e^{\sin^2 x}$

Solution:- $\frac{dy}{dx} = \frac{d}{dx} e^{\sin^2 x}$ $\left[\because \sin^2 x = (\sin x)^2\right]$

$$= e^{\sin^2 x} \times \frac{d}{dx}\sin^2 x.$$

$$= e^{\sin^2 x} \times 2\sin x \times \frac{d}{dx}\sin x$$

$$= e^{\sin^2 x} \times 2\sin x \times \cos x$$

Q-4 $y = [\tan(3x^2+5)]^5$

Solution:- $\frac{dy}{dx} = \frac{d}{dx}[\tan(3x^2+5)]^5$

$$= 5[\tan(3x^2+5)]^4 \times \frac{d}{dx}\tan(3x^2+5)$$

$$= 5[\tan(3x^2+5)]^4 \times \sec^2(3x^2+5) \times \frac{d}{dx}(3x^2+5)$$

$$= 5[\tan(3x^2+5)]^4 \times \sec^2(3x^2+5) \cdot (6x).$$

Shortcut-2 :-

Q:1 $y = \sqrt{\tan x}$ find $\frac{dy}{dx} = ?$

Solution :- $y = \sqrt{\tan x}$

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt{\tan x} \times \frac{d}{dx}\tan x$$

$$= \frac{1}{2\sqrt{\tan x}} \cdot \sec^2 x .$$

Q:-2 $y = \cos^2\sqrt{x}$ find $\frac{dy}{dx} = ?$

Solution :- $y = \cos^2\sqrt{x}$

$$= (\cos\sqrt{x})^2$$

Then $\frac{dy}{dx} = \frac{d}{dx}(\cos\sqrt{x})^2 \times \frac{d}{dx}\cos\sqrt{x} \times \frac{d}{dx}\sqrt{x}$

$$= 2\cos\sqrt{x} \times (-\sin\sqrt{x}) \times \frac{1}{2\sqrt{x}}$$

Q.3 $y = \sqrt{\sin\sqrt{x}}$

Solution :- Given $y = \sqrt{\sin\sqrt{x}}$

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt{\sin\sqrt{x}} \times \frac{d}{dx}\sin\sqrt{x} \times \frac{d}{dx}\sqrt{x}$$

$$= \frac{1}{2\sqrt{\sin\sqrt{x}}} \times \cos\sqrt{x} \times \frac{1}{2\sqrt{x}}$$ (Ans).

Some Imp Questions :-

→ find Derivative of the followings:

Q.1 $y = \log(\log(\log x))$

Solution :- $\frac{dy}{dx} = \frac{d}{dx}\log(\log(\log x))$

$$= \frac{1}{\log(\log x)} \times \frac{d}{dx}\log(\log x)$$

$$= \frac{1}{\log(\log x)} \times \frac{1}{\log x} \times \frac{d}{dx}\log x .$$

$$= \frac{1}{\log(\log x)} \times \frac{1}{\log x} \times \frac{1}{x}$$

Q.2 $y = \sqrt{e^{\sqrt{x}}}$

Solution :- $\frac{dy}{dx} = \frac{d}{dx}\sqrt{e^{\sqrt{x}}}$

$$= \frac{1}{2\sqrt{e^{\sqrt{x}}}} \times \frac{d}{dx}e^{\sqrt{x}}$$

$$= \frac{1}{2\sqrt{e^{\sqrt{x}}}} \times e^{\sqrt{x}} \times \frac{d}{dx}\sqrt{x}.$$

$$= \frac{1}{2\sqrt{e^{\sqrt{x}}}} \times e^{\sqrt{x}} \times \frac{1}{2\sqrt{x}}$$

Q.3 $y = \cos(\log x)^2$

Solution:- $\frac{dy}{dx} = \frac{d}{dx}\cos(\log x)^2$

$= -\sin(\log x)^2 \times \frac{d}{dx}(\log x)^2$

$= -\sin(\log x)^2 \times 2\log x \times \frac{d}{dx}\log x$

$= -\sin(\log x)^2 \times 2\log x \times \frac{1}{x}$

Q.4 $y = \log(x+\sqrt{x^2+a})$

Solution:- $\frac{dy}{dx} = \frac{d}{dx}\log(x+\sqrt{x^2+a})$

$= \frac{1}{x+\sqrt{x^2+a}} \frac{d}{dx}(x+\sqrt{x^2+a})$

$= \frac{1}{x+\sqrt{x^2+a}} \left(\frac{d}{dx}x + \frac{d}{dx}\sqrt{x^2+a}\right)$

$= \frac{1}{x+\sqrt{x^2+a}} \left[1 + \frac{1}{2\sqrt{x^2+a}} \times \frac{d}{dx}(x^2+a)\right]$

$= \frac{1}{x+\sqrt{x^2+a}} \left[1 + \frac{2x}{2\sqrt{x^2+a}}\right]$

$= \frac{1}{x+\sqrt{x^2+a}} \left[\frac{\sqrt{x^2+a}+x}{\sqrt{x^2+a}}\right]$

$= \frac{1}{\sqrt{x^2+a}}$

Derivative of Inverse Trigonometric functions by Trigonometrical Transformation:-

Imp Trigonometric Formulae:

(1) $\sin^2\theta + \cos^2\theta = 1$

(2) $\tan^2\theta + 1 = \sec^2\theta$

(3) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

(4) $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

or $2\sin\theta\cdot\cos\theta$

(5) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

or $2\cos^2\theta - 1$

or $1 - 2\sin^2\theta$

or $\frac{1-\tan^2\theta}{1+\tan^2\theta}$

(6) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(7) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(8) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

(9) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

(10) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\cdot\tan B}$

(11) $1 - \cos 2\theta = 2\sin^2\theta$

(12) $1 + \cos 2\theta = 2\cos^2\theta$

(13) $1 - \sin 2\theta = (\cos\theta - \sin\theta)^2$

(14) $1 + \sin 2\theta = (\cos\theta + \sin\theta)^2$

CASE-1 :-

Evaluate the derivative of the following functions

Q.1 $y = \tan^{-1} 2x$

Solution :- $\frac{dy}{dx} = \frac{d}{dx} \tan^{-1} 2x$

$= \frac{1}{1+(2x)^2} \times \frac{d}{dx} 2x$

$= \frac{1}{1+4x^2} \times 2$

Q.2 $y = \cos^{-1}(\cot x)$

Solution :- $\frac{dy}{dx} = \frac{d}{dx} \cos^{-1}(\cot x)$

$= \frac{-1}{\sqrt{1-(\cot x)^2}} \times \frac{d}{dx} \cot x$

$= \frac{-1}{\sqrt{1-\cot^2 x}} \cdot (-\operatorname{cosec}^2 x)$

$= \frac{\operatorname{cosec}^2 x}{\sqrt{1-\cot^2 x}}$

Q.3 $y = \sqrt{\sin^{-1}\sqrt{x}}$

Solution :- $\frac{dy}{dx} = \frac{d}{dx} \sqrt{\sin^{-1}\sqrt{x}}$

$= \frac{1}{2\sqrt{\sin^{-1}\sqrt{x}}} \times \frac{d}{dx} \sin^{-1}\sqrt{x}$

$= \frac{1}{2\sqrt{\sin^{-1}\sqrt{x}}} \times \frac{1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{d}{dx}\sqrt{x}$

$= \frac{1}{2\sqrt{\sin^{-1}\sqrt{x}}} \times \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$

Case-2 :-

Evaluate the derivative of the following function

Q.1 $y = \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

Solution :- ~~$\frac{dy}{dx} = \frac{d}{dx} \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$~~

Given $y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$

$= \tan^{-1}\sqrt{\frac{2\sin^2 x/2}{2\cos^2 x/2}}$

$= \tan^{-1}\sqrt{\tan^2 x/2}$

$= \tan^{-1}(\tan x/2)$

$= x/2$

Then $\frac{dy}{dx} = \frac{d}{dx}\frac{x}{2} = \frac{1}{2}$

Q.2 $y = \tan^{-1}\sqrt{\frac{1+\cos x}{1-\cos x}}$

Solution :- Given $y = \tan^{-1}\sqrt{\frac{1+\cos x}{1-\cos x}}$

$= \tan^{-1}\sqrt{\frac{2\cos^2 x/2}{2\sin^2 x/2}}$

$= \tan^{-1}\sqrt{\cot^2 x/2}$

$= \tan^{-1}(\cot x/2)$

$= \tan^{-1}(\tan(\pi/2 - x/2))$

$= \pi/2 - x/2$

Then $\frac{dy}{dx} = \frac{d}{dx}\left(\pi/2 - x/2\right)$

$= 0 - 1/2$

$= -1/2$

Q.3 $y = \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

Solution:- Given $y = \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

Dividing $\cos x$ both in N^r and D^r

$\Rightarrow y = \tan^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right)$

$= \tan^{-1}\left(\frac{\tan \pi/4 - \tan x}{1 + \tan\frac{\pi}{4}\cdot\tan x}\right)$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} - x\right)\right)$

$= \frac{\pi}{4} - x$

Then $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} - x\right)$

$= 0 - 1$

$= -1$

Q.4 $y = \tan^{-1}\sqrt{\frac{1+\sin x}{1-\sin x}}$

Given $y = \tan^{-1}\sqrt{\frac{1+\sin x}{1-\cos x}}$

$= \tan^{-1}\sqrt{\frac{(\cos x/2 + \sin x/2)^2}{(\cos x/2 - \sin x/2)^2}}$

$= \tan^{-1}\left(\frac{\cos x/2 + \sin x/2}{\cos x/2 - \sin x/2}\right)$

Dividing $\cos x/2$ both in N^r and D^r

$= \tan^{-1}\left(\frac{1 + \tan x/2}{1 - \tan x/2}\right)$

$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} + \tan x/2}{1 - \tan\frac{\pi}{4}\cdot\tan x/2}\right)$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + x/2\right)\right)$

$= \frac{\pi}{4} + x/2$

Then $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} + x/2\right)$

$= 0 + \frac{1}{2}$

$= \frac{1}{2}$

Q.5 $y = \tan^{-1}(\operatorname{cosec} x + \cot x)$

Given $y = \tan^{-1}(\operatorname{cosec} x + \cot x)$

$= \tan^{-1}\left(\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right)$

$= \tan^{-1}\left(\frac{1 + \cos x}{\sin x}\right)$

$= \tan^{-1}\left(\frac{2\cos^2 x/2}{2\sin x/2 \cdot \cos x/2}\right)$

$= \tan^{-1}\left(\frac{\cos x/2}{\sin x/2}\right)$

$= \tan^{-1}\left(\cot x/2\right)$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{2} - x/2\right)\right)$

$= \pi/2 - x/2$

$\frac{dy}{dx} = \frac{d}{dx}\left(\pi/2 - x/2\right)$

$= 0 - 1/2$

$= -1/2$

Case-3 :-

Evaluate the derivative of the followings :

Q-1 $y = \sin^{-1}(3x - 4x^3)$

Solution Given $y = \sin^{-1}(3x - 4x^3)$

put $x = \sin\theta$

$\Rightarrow y = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$= \sin^{-1}(\sin 3\theta)$

$= 3\theta$

$\Rightarrow y = 3\sin^{-1}x$ $\left(\begin{array}{l}\text{as } x = \sin\theta \\ \Rightarrow \theta = \sin^{-1}x\end{array}\right)$

$\frac{dy}{dx} = \frac{d}{dx} 3\sin^{-1}x$

$= 3\left(\frac{1}{\sqrt{1-x^2}}\right)$

$= \frac{3}{\sqrt{1-x^2}}$

Q-2 $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution :- Given $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

put $x = \tan\theta$

$\Rightarrow y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}(\cos 2\theta)$

$= 2\theta$

$= 2\tan^{-1}x$ $\left(\begin{array}{l}\text{as } x = \tan\theta \\ \Rightarrow \theta = \tan^{-1}x\end{array}\right)$

$\frac{dy}{dx} = \frac{d}{dx} 2\tan^{-1}x$

$= 2\left(\frac{1}{1+x^2}\right)$

$= \frac{2}{1+x^2}$

Q.3 $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Solution:- Given $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

put $x = \tan\theta$

$$\Rightarrow y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\theta/2\cdot\cos\theta/2}\right)$$

$$= \tan^{-1}\left(\tan\theta/2\right)$$

$$= \theta/2$$

$$= \frac{\tan^{-1}x}{2}$$

$$\frac{dy}{dx} = \frac{d}{dx}\frac{\tan^{-1}x}{2}$$

$$= \frac{1}{2}\left(\frac{1}{1+x^2}\right)$$

Derivative of Parametric functions:-

Parametric function:-

In parametric function both x and y are given as functions of another variable, called a parameter.

→ **Method to find $\frac{dy}{dx}$ when x and y are functions of t**

let $x = f(t)$ and $y = g(t)$

$$\text{then } \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

→ **Method to find $\frac{dy}{dx}$ when x and y are functions of θ**

let $x = f(\theta)$ and $y = g(\theta)$

$$\text{then } \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

Q.-1 find $\frac{dy}{dx}$ for the following functions:

(i) If $x = at^2$ and $y = 2bt$.

Solution: Given $x = at^2$

$$\frac{dx}{dt} = \frac{d}{dt}at^2$$

$$= a\frac{d}{dt}t^2$$

$$= a(2t)$$

$$= 2at$$

$y = 2bt$

$\frac{dy}{dt} = \frac{d}{dt}2bt$

$= 2b\left(\frac{d}{dt}t\right)$

$= 2b$

Then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2b}{2at} = \frac{b}{at}$

(ii) $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$

Solution:-

Given $x = a(\theta + \sin\theta)$

$\frac{dx}{d\theta} = \frac{d}{d\theta}a(\theta+\sin\theta)$

$= a\left[\frac{d}{d\theta}\theta + \frac{d}{d\theta}\sin\theta\right]$

$= a[1+\cos\theta]$

$y = a(1-\cos\theta)$

$\frac{dy}{d\theta} = \frac{d}{d\theta}a(1-\cos\theta)$

$= a\left[\frac{d}{d\theta}1 - \frac{d}{d\theta}\cos\theta\right]$

$= a(0-(-\sin\theta))$

$= a\sin\theta$

Then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$= \frac{a\sin\theta}{a(1+\cos\theta)}$

$= \frac{\sin\theta}{1+\cos\theta}$

Derivative of a function w.r.t another function

Suppose we have to differentiate $f(x)$ w.r.t $g(x)$

In this case let $y = f(x)$

and $z = g(x)$

i.e. above becomes a parametric function with parameter 'x'.

Then $\boxed{\frac{dy}{dz} = \frac{dy/dx}{dz/dx}}$

Q:-1 Differentiate $\sin^{-1}x$ w.r.t. $\cos^{-1}x$

Solution:- Let $y = \sin^{-1}x$ and $z = \cos^{-1}x$

$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}x$

$= \frac{1}{\sqrt{1-x^2}}$

$\frac{dz}{dx} = \frac{d}{dx}\cos^{-1}x$

$= \frac{-1}{\sqrt{1-x^2}}$

Then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{1/\sqrt{1-x^2}}{-1/\sqrt{1-x^2}} = -1$

Q:-2 Differentiate $\sqrt{x}$ w.r.t x^2

Let $y = \sqrt{x}$ and $z = x^2$

$\frac{dy}{dx} = \frac{d}{dx}\sqrt{x}$
$= \frac{1}{2\sqrt{x}}$

$\frac{dz}{dx} = \frac{d}{dx}x^2$
$= 2x$

Then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{1/2\sqrt{x}}{2x} = \frac{1}{4x\sqrt{x}}$

Q.3 Differentiate $\sin^2 x$ w.r.t. $(\ln x)^2$

Solution :- let $y = \sin^2 x$ and $z = (\ln x)^2$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\sin^2 x$
$= 2\sin x \cdot \frac{d}{dx}\sin x$
$= 2\sin x \cdot \cos x$
$= \sin 2x$

$\Rightarrow \frac{dz}{dx} = \frac{d}{dx}(\ln x)^2$
$= 2\ln x \frac{d}{dx}\ln x$
$= 2\ln x \cdot \left(\frac{1}{x}\right)$

Then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{\sin 2x}{2\ln x\,(1/x)}$

Logarithmic Differentiation :-

To find derivative of a function power another function (i.e. $f(x)^{g(x)}$), Logarithmic differentiation is helpful.

Methods to follow:

Step1 Given $y = f(x)^{g(x)}$

Step2 Take Logarithmic of the function on both sides. i.e. $\log y = \log f(x)^{g(x)}$

Step3 use the formula $\log x^n = n \log x$
i.e. $\log y = g(x) \cdot \log f(x)$

Step4 Differentiate both sides.
i.e. $\frac{d}{dx}\log y = \frac{d}{dx}\{g(x) \cdot \log f(x)\}$

Q:- Find $\frac{dy}{dx}$ of the followings:

(i) $y = x^x$

Solution :- Given $y = x^x$

Take Logarithm on both sides.

$\Rightarrow \log y = \log x^x$

$= x \times \log x$

Differentiate both sides w.r.t x

$\Rightarrow \frac{d}{dx} \log y = \frac{d}{dx} \{x \times \log x\}$

$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \left(\frac{d}{dx} x\right) \log x + x \left(\frac{d}{dx} \log x\right)$

$= (1) \log x + x \left(\frac{1}{x}\right)$

$= \log x + 1$

$\Rightarrow \frac{dy}{dx} = y [\log x + 1]$

$= x^x [\log x + 1]$

(ii) $(\sin x)^{\log x}$

Solution:- Let $y = (\sin x)^{\log x}$

Take Log on both sides.

$\Rightarrow \log y = \log (\sin x)^{\log x}$

$= \log x \times \log (\sin x)$

Differentiate both sides w.r.t x

$\Rightarrow \frac{d}{dx} \log y = \frac{d}{dx} [\log x \times \log \sin x]$

$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \left(\frac{d}{dx} \log x\right) \log \sin x + \log x \left(\frac{d}{dx} \log \sin x\right)$

$= \left(\frac{1}{x}\right) \log \sin x + \log x \left(\frac{1}{\sin x} \cdot \cos x\right)$

$= \frac{\log \sin x}{x} + \log x \cdot \cot x$

$\Rightarrow \frac{dy}{dx} = y \left\{ \frac{\log \sin x}{x} + \log x \cdot \cot x \right\}$

$= \sin x^{\log x} \left\{ \frac{\log \sin x}{x} + \log x \cdot \cot x \right\}$

(iii) Differentiate $x^{\sin^{-1} x} + (\sin^{-1} x)^x$

Solution:- Given $y = x^{\sin^{-1} x} + (\sin^{-1} x)^x$

let $u = x^{\sin^{-1} x}$

$v = (\sin^{-1} x)^x$

Then $y = u + v$

Diff both sides w.r.t x

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ —— (1)

consider $u = x^{\sin^{-1}x}$

Taking Log on both sides.

$$\log u = \log x^{\sin^{-1}x}$$

$$= \sin^{-1}x \,(\log x)$$

Diff. bothsides w.r.t x

$$\Rightarrow \frac{d}{dx}\log u = \frac{d}{dx}\{\sin^{-1}x \cdot \log x\}$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \left(\frac{d}{dx}\sin^{-1}x\right)\log x + \sin^{-1}x\left(\frac{d}{dx}\log x\right)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}\log x + \sin^{-1}x\left(\frac{1}{x}\right).$$

$$\Rightarrow \frac{du}{dx} = u\left[\frac{\log x}{\sqrt{1-x^2}} + \frac{\sin^{-1}x}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^{\sin^{-1}x}\left[\frac{\log x}{\sqrt{1-x^2}} + \frac{\sin^{-1}x}{x}\right] \text{———— (ii)}$$

Again consider $v = (\sin^{-1}x)^x$

Taking Log on bothsides.

$$\log v = \log(\sin^{-1}x)^x$$

$$= x \log(\sin^{-1}x)$$

Differentiate bothsides w.r.t x

$$\Rightarrow \frac{d}{dx}\log v = \frac{d}{dx}\{x \log(\sin^{-1}x)\}$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left(\frac{d}{dx}x\right)\log\sin^{-1}x + x\left(\frac{d}{dx}\log\sin^{-1}x\right)$$

$$= \log\sin^{-1}x + x\frac{1}{\sin^{-1}x}\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dv}{dx} = v\left[\log\sin^{-1}x + \frac{x}{\sin^{-1}x}\frac{1}{\sqrt{1-x^2}}\right]$$

$$= (\sin^{-1}x)^x\left[\log\sin^{-1}x + \frac{x}{\sqrt{1-x^2}\,\sin^{-1}x}\right]$$

Derivative of implicity function

Definition of implicity function:-

An eqⁿ of the form $f(x,y)=0$ in which y cann't be directly expressed in terms of x known as implicit function of x and y.

Q.-1 Find $\frac{dy}{dx}$, when $x^2+y^2=2axy$

Solution:- Given $x^2+y^2=2axy$

Diff. both sides w.r.t x

$$\Rightarrow \frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(2axy)$$

$$\Rightarrow \frac{d}{dx}x^2+\frac{d}{dx}y^2=\frac{d}{dx}(2axy)$$

$$\Rightarrow 2x+2y\frac{dy}{dx}=2a\frac{d}{dx}(xy)$$

$$=2a\left[\left(\frac{d}{dx}x\right)y+x\left(\frac{d}{dx}y\right)\right]$$

$$\Rightarrow 2x+2y\frac{dy}{dx}=2a\left[y+x\frac{dy}{dx}\right]$$

$$=2ay+2ax\frac{dy}{dx}$$

$$\Rightarrow 2y\frac{dy}{dx}-2ax\frac{dy}{dx}=2ay-2x$$

$$\Rightarrow [2y-2ax]\frac{dy}{dx}=2ay-2x$$

$$\Rightarrow \frac{dy}{dx}=\frac{2ay-2x}{2y-2ax}$$

Q.2 Find $\frac{dy}{dx}$, where $\cos(x+y)=y\sin x$

Solution:- Given $\cos(x+y)=y\sin x$

Diff. both sides w.r.t x

$$\Rightarrow \frac{d}{dx}\cos(x+y)=\frac{d}{dx}y\sin x$$

$$\Rightarrow -\sin(x+y)\frac{d}{dx}(x+y)=\left(\frac{d}{dx}y\right)\sin x+y\left(\frac{d}{dx}\sin x\right)$$

$$\Rightarrow -\sin(x+y)\left\{\frac{d}{dx}x+\frac{d}{dx}y\right\}=\frac{dy}{dx}\sin x+y\cos x$$

$$\Rightarrow -\sin(x+y)\left\{1+\frac{dy}{dx}\right\}=\sin x\frac{dy}{dx}+y\cos x$$

$$\Rightarrow -\sin(x+y)-\sin(x+y)\frac{dy}{dx}=\sin x\frac{dy}{dx}+y\cos x$$

$$\Rightarrow \sin x\frac{dy}{dx}+\sin(x+y)\frac{dy}{dx}=-y\cos x-\sin(x+y)$$

$$\Rightarrow [\sin x+\sin(x+y)]\frac{dy}{dx}=-[y\cos x+\sin(x+y)]$$

$$\Rightarrow \frac{dy}{dx}=-\frac{y\cos x+\sin(x+y)}{\sin x+\sin(x+y)}$$

Q.3 Differentiate $x^y = y^x$

Solution:- Given $x^y = y^x$

Taking Log on both sides.

$\log x^y = \log y^x$

$\Rightarrow y \cdot \log x = x \times \log y$

Diff. both sides w.r.t x

$\Rightarrow \frac{d}{dx}\{y \times \log x\} = \frac{d}{dx}\{x \times \log y\}$

$\Rightarrow \left(\frac{d}{dx} y\right) \log x + y\left(\frac{d}{dx}\log x\right) = \left(\frac{d}{dx}x\right)\log y + x\left(\frac{d}{dx}\log y\right)$

$\Rightarrow \frac{dy}{dx} \log x + y\left(\frac{1}{x}\right) = \log y + x \frac{1}{y}\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} \log x - \frac{x}{y}\frac{dy}{dx} = \log y - \frac{y}{x}$

$\Rightarrow \frac{dy}{dx}\left[\log x - \frac{x}{y}\right] = \log y - y/x$

$\Rightarrow \frac{dy}{dx} = \frac{\log y - y/x}{\log x - x/y}$

Q.4:- If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, P.T. $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

Given $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

put $x = \sin\alpha$ then $\alpha = \sin^{-1}x$

$y = \sin\beta$ $\beta = \sin^{-1}y$

$\Rightarrow \sqrt{1-\sin^2\alpha} + \sqrt{1-\sin^2\beta} = a(\sin\alpha - \sin\beta)$

$\Rightarrow \sqrt{\cos^2\alpha} + \sqrt{\cos^2\beta} = a(\sin\alpha - \sin\beta)$

$\Rightarrow \frac{\cos\alpha + \cos\beta}{\sin\alpha - \sin\beta} = a$

$\Rightarrow \frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha-\beta}{2}\right)} = a$

$\Rightarrow \cot\left(\frac{\alpha-\beta}{2}\right) = a$

$\Rightarrow \frac{\alpha-\beta}{2} = \cot^{-1}a$

$\Rightarrow \alpha - \beta = 2\cot^{-1}a$

$\Rightarrow \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a$

Diff. w.r.t 'x'

$\Rightarrow \frac{d}{dx}\sin^{-1}x - \frac{d}{dx}\sin^{-1}y = \frac{d}{dx}2\cot^{-1}a$

$\Rightarrow \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$

$\Rightarrow \frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ (proved).

-: Successive Differentiation :-

Let $y = f(x)$ be the function, then its derivative w.r.t x is denoted by $\frac{dy}{dx}$ / y' / y_1 / $f'(x)$

Which is known as derivative of first order.

Now Successive differentiation means again and again differentiation upto 'n' no. of times.

* Successive diff. upto 2 no. of time.

Let $y = f(x)$

its first derivative is $\frac{dy}{dx}$ (Known as first order derivative)

if we again differentiate w.r.t 'x'

i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ (Known as second order derivative)

Notations of 2nd order derivative:

y'' / $f''(x)$ / y_2 / $\frac{d^2y}{dx^2}$

where $\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)}$

Q.1 Find y_1 and y_2 of the followings:

i) $y = \log x$

Solution $y_1 = \frac{d}{dx}(\log x) = \frac{1}{x}$

$y_2 = \frac{d}{dx}(y_1) = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

(ii) $y = \ln(\sin x)$

Solution $y_1 = \frac{d}{dx}\{\ln(\sin x)\} = \frac{1}{\sin x}\cdot\cos x = \cot x$

$y_2 = \frac{d}{dx}(y_1) = \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

Q.2 find $\frac{d^2y}{dx^2}$ of the followings :-

(i) $x = at^2$, $y = 2at$ find $\frac{d^2y}{dx^2}$.

Solution:- Given $x = at^2$, $y = 2at$

$\frac{dx}{dt} = \frac{d}{dt}at^2$ $\quad$ $\frac{dy}{dt} = \frac{d}{dt}2at$

$= a\frac{d}{dt}t^2$ $\quad$ $= 2a\frac{d}{dt}t$

$= 2at$ $\quad$ $= 2a$

Then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$

(Const.) $\frac{1}{dt}$

Then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{t}\right) = \left(-\frac{1}{t^2}\right)\cdot\frac{dt}{dx}$

$= -\frac{1}{t^2}\times\frac{1}{\frac{dx}{dt}}$

$= -\frac{1}{t^2} \cdot \frac{1}{2at}$

$= -\frac{1}{2at^3}$ (Ans).

(ii) $x = a\cos^3\theta$, $y = a\sin^3\theta$, find $\frac{d^2y}{dx^2}$.

Solution :- Given $x = a\cos^3\theta$

$\frac{dx}{d\theta} = \frac{d}{d\theta} a\cos^3\theta$

$= 3a\cos^2\theta \cdot (-\sin\theta)$

$= -3a\cos^2\theta \cdot \sin\theta$

Again $y = a\sin^3\theta$

$\frac{dy}{d\theta} = \frac{d}{d\theta} a\sin^3\theta$

$= 3a\sin^2\theta \cdot \cos\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta \cdot \cos\theta}{-3a\cos^2\theta \cdot \sin\theta}$

$= -\frac{\sin\theta}{\cos\theta} = -\tan\theta$

Then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-\tan\theta)$

$= -\sec^2\theta \frac{d\theta}{dx}$

$= -\sec^2\theta \cdot \frac{1}{\frac{dx}{d\theta}}$

$= \frac{-\sec^2\theta}{-3a\cos^2\theta \cdot \sin\theta} = \frac{1}{3a\cos^4\theta \cdot \sin\theta}$

Q:-3 (i) If $y = A\cos x + B\sin x$ then P.T. $\frac{d^2y}{dx^2} + y = 0$

Solution :- Given $y = A\cos x + B\sin x$.

$\frac{dy}{dx} = \frac{d}{dx}(A\cos x + B\sin x)$

$= A(-\sin x) + B\cos x$

$= -A\sin x + B\cos x$.

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-A\sin x + B\cos x)$

$= -A\cos x + B(-\sin x)$

$= -A\cos x - B\sin x$

$= -(A\cos x + B\sin x)$

$\Rightarrow \frac{d^2y}{dx^2} = -y$

$\Rightarrow \frac{d^2y}{dx^2} + y = 0$ (proved)

(ii) If $y = \tan^{-1}x$, P.T. $(1+x^2)y_2 + 2xy_1 = 0$

Given $y = \tan^{-1}x$

Then $\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$\Rightarrow (1+x^2)\frac{dy}{dx} = 1$

Again diff. both sides w.r.t x

$$\frac{d}{dx}\left\{\frac{dy}{dx}\cdot(1+x^2)\right\} = \frac{d}{dx}(1)$$

$$\Rightarrow \left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)\right\}(1+x^2) + \frac{dy}{dx}\left\{\frac{d}{dx}(1+x^2)\right\} = 0$$

$$\Rightarrow \frac{d^2y}{dx^2}(1+x^2) + \frac{dy}{dx}(2x) = 0$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0 \qquad \left(\because y_2 = \frac{d^2y}{dx^2}\right.$$

or $(1+x^2)y_2 + 2xy_1 = 0$ (proved) $\left(y_1 = \frac{dy}{dx}\right)$

Q.4 (i) If $y = e^{m\cos^{-1}x}$ P.T. $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$

Given $y = e^{m\cos^{-1}x}$

$$\frac{dy}{dx} = \frac{d}{dx}e^{m\cos^{-1}x}$$

$$= e^{m\cos^{-1}x}\cdot\frac{d}{dx}m\cos^{-1}x$$

$$= e^{m\cos^{-1}x}\cdot\left(\frac{-m}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = -m\,e^{m\cos^{-1}x}$$

$$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = -my \quad \text{——— (1)}$$

~~Again differentiating both sides w.r.t 'x'.~~

Squaring both sides.

$\Rightarrow \left(\sqrt{1-x^2}\right)^2\left(\frac{dy}{dx}\right)^2 = (-my)^2$

$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2y^2$

Now diff. both sides w.r.t x.

$\Rightarrow \frac{d}{dx}\left\{(1-x^2)\left(\frac{dy}{dx}\right)^2\right\} = \frac{d}{dx}(m^2y^2)$

$\Rightarrow \left\{\frac{d}{dx}(1-x^2)\right\}\left(\frac{dy}{dx}\right)^2 + (1-x^2)\left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)^2\right\} = m^2\frac{d}{dx}y^2$

$\Rightarrow (-2x)\left(\frac{dy}{dx}\right)^2 + (1-x^2)\,2\frac{dy}{dx}\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) = m^2\,2y\frac{dy}{dx}$

$\Rightarrow -2x\left(\frac{dy}{dx}\right)^2 + 2(1-x^2)\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = m^2y\left(2\frac{dy}{dx}\right)$

$\Rightarrow 2\frac{dy}{dx}\left\{-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2}\right\} = m^2y\left(2\frac{dy}{dx}\right)$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = m^2y$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$ (proved)

(iii) If $x = \sin t$, $y = \sin(pt)$ then show that

$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$

Solution :- Given $y = \sin(pt)$

$= \sin(p\sin^{-1}x)$ $(\because x = \sin t)$.

Then $\frac{dy}{dx} = \cos(p\sin^{-1}x)\cdot\frac{p}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = p\cos(p\sin^{-1}x)$

Squaring both the sides.

$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = p^2\cos^2(p\sin^{-1}x)$

$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = p^2\{1-\sin^2(p\sin^{-1}x)\}$

$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = p^2 - p^2\sin^2(p\sin^{-1}x)$

$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = p^2 - p^2y^2$ $(\because y = \sin(p\sin^{-1}x))$

Again Diff. we get

$\Rightarrow \frac{d}{dx}\left\{(1-x^2)\left(\frac{dy}{dx}\right)^2\right\} = \frac{d}{dx}\{p^2 - p^2y^2\}$

$\Rightarrow \left\{\frac{d}{dx}(1-x^2)\right\}\left(\frac{dy}{dx}\right)^2 + (1-x^2)\left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)^2\right\} = -p^2\cdot 2y\frac{dy}{dx}$

$\Rightarrow -2x\left(\frac{dy}{dx}\right)^2 + (1-x^2)\,2\frac{dy}{dx}\frac{d^2y}{dx^2} = -p^2\,2y\frac{dy}{dx}$

$\Rightarrow 2\frac{dy}{dx}\left[-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2}\right] = 2\frac{dy}{dx}(-p^2y)$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -p^2y$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$ (proved)

-: Partial Differentiation :-

Partial Differentiation means derivative of a function of several variables (functions dependent on two or more variables)

For example :- (i) $y = x^2t + x^3t^2$

here y is a function of two variables x & t

or $y = f(x, t)$

(ii) $z = x^2y + xy^2$

here z is a function of two variables x and y

i.e. $z = f(x, y)$

where x and y are independent variables.

and z is dependent variable.

(iii) $u = xyz + x^3 + y^3 + z^3$

here u is a function of three variables. x, y, z

where x, y, z are independent variable.

and u is dependent variable.

And Partial differentiation is used to evaluate the derivative of these type of functions.

Methodology :-

Given $z = f(x, y)$

then its partial derivative w.r.t 'x' is denoted as

$\frac{\partial z}{\partial x}$ or f_x (treating y as constant).

and partial derivative w.r.t 'y' is denoted as

$\frac{\partial z}{\partial y}$ or f_y (treating x as constant).

Q.1 $z = x^2y + xy^2$ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Sol^n $z = x^2y + xy^2$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2y + xy^2)$$

$$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial x}(xy^2)$$

$$= 2xy + y^2$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2y + xy^2)$$

$$= \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial y}(xy^2)$$

$$= x^2 + 2xy$$

Homogeneous function:-

Defⁿ A function $f(x,y)$ is said to be homogeneous in x and y of degree 'n'

if $f(tx, ty) = t^n f(x,y)$

or A function $f(x,y)$ is said to be homogeneous in x and y of degree n if sum of all powers of x and y is equal to n in each term.

For example:-

Check whether $f(x,y) = x^4 + x^3y - y^4$ is homogeneous or not?

1st method $f(x,y) = x^4 + x^3y - y^4$

$$f(tx, ty) = (tx)^4 + (tx)^3(ty) - (ty)^4$$

$$= t^4x^4 + t^3x^3ty - t^4y^4$$

$$= t^4x^4 + t^4x^3y - t^4y^4$$

$$= t^4(x^4 + x^3y - y^4)$$

$$= t^4 f(x,y)$$

So $f(x,y)$ is a homogeneous function of degree 4.

2nd method

$$f(x,y) = x^4 + x^3y - y^4$$

Here each term i.e. 1st term x^4 (degree 4)

2nd term x^3y (sum of power is 4)

3rd term y^4 (Power 4).

So $f(x,y)$ is a homogeneous function of degree 4.

Euler's Theorem:-

If z is a homogeneous function of degree n then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

Integration

Standard formulas:

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

2. $\int e^x \, dx = e^x + c$

3. $\int a^x \, dx = \frac{a^x}{\log a} + c$

4. $\int \frac{1}{x} \, dx = \log x + c$

5. $\int k \, dx = kx + c$

6. $\int \sin x \, dx = -\cos x + c$

7. $\int \cos x \, dx = \sin x + c$

8. $\int \sec^2 x \, dx = \tan x + c$

9. $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

10. $\int \sec x \cdot \tan x \, dx = \sec x + c$

11. $\int \operatorname{cosec} x \cdot \cot x = -\operatorname{cosec} x + c$

(12) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}x + c$

or $-\cos^{-1}x + c$

(13) $\int \frac{1}{1+x^2} dx = \tan^{-1}x + c$

or $-\cot^{-1}x + c$

(14) $\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1}x + c$

or $-\operatorname{cosec}^{-1}x + c$

— o — o — o —

Integration formulas derived from Substitution method

(1) $\int \tan x \, dx = \log|\sec x| + c$

or $-\log|\cos x| + c$.

(2) $\int \cot x \, dx = \log|\sin x| + c$

(3) $\int \sec x \, dx = \log|\sec x + \tan x| + c$

(4) $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + c$

Q:-1 (i) $\int \tan^2 x \, dx$

$= \int (\sec^2 x - 1)\, dx$

$= \tan x - x + c$

(ii) $\int \sqrt{1-\sin 2x}\, dx$

$= \int \sqrt{(\cos x - \sin x)^2}\, dx$

$= \int (\cos x - \sin x)\, dx$

$= \sin x + \cos x + c$

(iii) $\int \frac{1}{\sin^2 x \cdot \cos^2 x}\, dx$

$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x}\, dx$

$= \int \frac{\sin^2 x}{\sin^2 x \cdot \cos^2 x}\, dx + \int \frac{\cos^2 x}{\sin^2 x \cdot \cos^2 x}\, dx$

$= \int \frac{1}{\cos^2 x}\, dx + \int \frac{1}{\sin^2 x}\, dx$

$= \int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx$

$= \tan x - \cot x + c$

Integration by ~~...~~
Substitution Method

Type I $\int f(ax+b)\,dx$

Take $ax+b=t$ $\Rightarrow a\,dx = dt$ $\Rightarrow dx = \frac{dt}{a}$

Then $\int f(ax+b)\,dx = \int f(t)\,\frac{dt}{a}$

Ex $\int \cos 3x\,dx.$

let $3x = t$

$\Rightarrow 3 = \frac{dt}{dx}$

$\Rightarrow dx = \frac{dt}{3}$

$= \int \cos t\,\frac{dt}{3}$

$= \frac{1}{3}\int \cos t\,dt$

$= \frac{1}{3}\sin t + c$

$= \frac{1}{3}\sin 3x + c$

Type II $\int f(g(x)) \cdot g'(x)\, dx$.

$$\text{let } g(x) = t$$
$$\Rightarrow g'(x) = \frac{dt}{dx}$$
$$\Rightarrow g'(x)\, dx = dt$$

Then $\int f(g(x)) \cdot g'(x)\, dx$

$= \int f(t)\, dt$

Ex:- $\int e^{\tan x} \cdot \sec^2 x\, dx$.

let $\tan x = t$

$\sec^2 x\, dx = dt$

$= \int e^t\, dt$

$= e^t + c$

$= e^{\tan x} + c$

Type III $\int \frac{f'(x)}{f(x)}\,dx$

$$\boxed{\begin{aligned} &\text{let } f(x) = t \\ &f'(x)\,dx = dt. \end{aligned}}$$

Then $\int \frac{f'(x)}{f(x)}\,dx = \int \frac{1}{t}\,dt$

Ex:- $\int \frac{\cos x}{\sin x}\,dx$

let $\sin x = t$

$\cos x\,dx = dt$.

$= \int \frac{1}{t}\,dt$

$= \log|t| + c$

$= \log|\sin x| + c$

Type IV $\int x^{n-1} f(x^n)\,dx$.

$$\boxed{\begin{aligned} &\text{let } x^n = t \\ &\Rightarrow n x^{n-1} = \frac{dt}{dx} \\ &\Rightarrow x^{n-1}\,dx = \frac{dt}{n} \end{aligned}}$$

Ex:- $\int x^6 \operatorname{cosec}^2(x^7)\, dx$.

let $x^7 = t$

$\Rightarrow 7x^6 = \frac{dt}{dx}$

$\Rightarrow x^6\, dx = \frac{dt}{7}$

$= \int \operatorname{cosec}^2 t \, \frac{dt}{7}$

$= \frac{1}{7} \int \operatorname{cosec}^2 t \, dt$

$= \frac{1}{7} (-\cot t) + c$

$= -\frac{1}{7} \cot x^7 + c$

Type V $\int [f(x)]^n f'(x)\, dx$.

let $f(x) = t$
$f'(x)\, dx = dt$

$= \int t^n \, dt$.

Ex:- $\int \cos^3 x \cdot \sin x \, dx$.

let $\cos x = t$

$-\sin x \, dx = dt$

$\sin x \, dx = -dt$

$= \int t^3 (-dt)$

$= -\int t^3 \, dt$

$= -\frac{t^4}{4} + c$

$= -\frac{\cos^4 x}{4} + c$

SPECIAL CASE

Q:-1 (i) $\int \sin^4 x \cdot \cos^3 x \, dx$.

$= \int \sin^4 x \cdot \cos^2 x \cdot \cos x \, dx$.

$= \int \sin^4 x \cdot (1 - \sin^2 x) \cos x \, dx$.

let $\sin x = t$

$\cos x \, dx = dt$

$= \int t^4(1-t^2)\,dt$

$= \int t^4 - t^6\,dt$

$= \frac{t^5}{5} - \frac{t^7}{7} + c$

$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$

(ii) $\int \cot^3 x \cdot \operatorname{cosec}^{16} x\,dx$

$= \sout{\int \cot^2 x \cdot \cos^{15} x} = \int \cot^2 x \cdot \operatorname{cosec}^{15} x \cdot \cot x \operatorname{cosec} x\,dx$

$= \int (\operatorname{cosec}^2 x - 1) \operatorname{cosec}^{15} x \cdot \cot x \cdot \operatorname{cosec} x\,dx$

let $\operatorname{cosec} x = t$

$\Rightarrow -\cot x \cdot \operatorname{cosec} x\,dx = dt$

$\Rightarrow \cot x \cdot \operatorname{cosec} x\,dx = -dt$

$= \int (t^2-1)t^{15}\,(-dt)$

$= -\int t^{17} - t^{15}\,dt$

$= -\left(\frac{t^{18}}{18} - \frac{t^{16}}{16}\right) + c = \frac{\operatorname{cosec}^{16} x}{16} - \frac{\operatorname{cosec}^{18} x}{18} + c$

(iii) $\int \cos 3x \cdot \sin 2x \, dx$.

$= \frac{1}{2} \int 2 \cos 3x \cdot \sin 2x \, dx$.

$= \frac{1}{2} \int \{ \sin(3x+2x) - \sin(3x-2x) \} \, dx$.

$= \frac{1}{2} \int \sin 5x - \sin x \, dx$.

$= \frac{1}{2} \left(\frac{-\cos 5x}{5} + \cos x \right) + c$

Integration by Trigometric Substitution

① $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + c$

② $\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

③ $\int \frac{1}{|x| \sqrt{x^2 - a^2}} \, dx = \frac{1}{a} \sec^{-1} \frac{x}{a} + c$

④ $\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \log \left| x + \sqrt{x^2 + a^2} \right| + c$

⑤ $\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \log \left| x + \sqrt{x^2 - a^2} \right| + c$

(6) $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

(7) $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

~~(8) $\int \sqrt{a^2-x^2}\,dx =$~~

(8) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(9) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$

(10) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$

Q.1 (i) $\int \frac{\cos x\,dx}{\sin^2 x+4}$

let $\sin x = t$

$\cos x\,dx = dt$

$= \int \frac{dt}{t^2+(2)^2}$

$= \frac{1}{2}\tan^{-1}\frac{t}{2} + c$

$= \frac{1}{2}\tan^{-1}\frac{\sin x}{2} + c$

(ii) $\int \frac{\cos x \, dx}{\sin^2 x \sqrt{\operatorname{cosec}^2 x - 4}}$

$= \int \frac{\cos x \, dx}{\sin x \cdot \sin x \sqrt{\operatorname{cosec}^2 x - 4}}$

$= \int \frac{\cot x \cdot \operatorname{cosec} x \, dx}{\sqrt{\operatorname{cosec}^2 x - 4}}$

let $\operatorname{cosec} x = t$

$\Rightarrow -\operatorname{cosec} x \cdot \cot x = \frac{dt}{dx}$

$\Rightarrow \operatorname{cosec} x \cdot \cot x \, dx = -dt$

$= \int \frac{-dt}{\sqrt{t^2 - (2)^2}}$

$= -\log \left| t + \sqrt{t^2 - (2)^2} \right| + c$

$= -\log \left| \operatorname{cosec} x + \sqrt{\operatorname{cosec}^2 x - 4} \right| + c$

SPECIAL CASE

Case I $\int \sqrt{ax^2+bx+c}\, dx$.

or $\int \frac{\text{const}}{\sqrt{ax^2+bx+c}}\, dx$ or $\int \frac{\text{const}}{ax^2+bx+c}\, dx$.

Then convert ax^2+bx+c into Perfect square

Q.1 $\int \frac{dx}{x^2+6x+13}$

$= \int \frac{dx}{(x)^2+2\cdot x\cdot 3+(3)^2-(3)^2+13}$

$= \int \frac{dx}{(x+3)^2+4}$

let $x+3=t$

$dx=dt$

$= \int \frac{dt}{t^2+(2)^2}$

$= \frac{1}{2}\tan^{-1}\frac{t}{2}+C$

$= \frac{1}{2}\tan^{-1}\frac{x+3}{2}+C$

Case II

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$$

or $\int (px+q)\sqrt{ax^2+bx+c}\,dx$ . . .

Then $\boxed{\text{let } ax^2+bx+c=t}$

Integration Byparts

Imp

$\int (\text{1st fun})(\text{2nd fun})\,dx$

$$= \text{1st fun} \int \text{2nd fun}\,dx - \int \left[\left(\frac{d}{dx}\,\text{1st fun}\right)\left(\int \text{2nd fun}\,dx\right)\right] dx$$

How to choose 1st & 2nd functn.

I L A T E

- I → Inverse Trigo functn.
- L → Logarithmic funct.
- A → Algebraic fun.
- T → Trigonometric funn
- E → Exponential fun.

Q.1 Evaluate

$$\int \cos x \cdot x \, dx.$$

I L A T E

A ↓ x, T ↓ $\cos x$.

1^{st} fun $= x$

2^{nd} " $= \cos x$.

Solution

$$\int \cos x \cdot x \, dx = x \int \cos x \, dx - \int \left[\left(\frac{d}{dx} x \right) \left(\int \cos x \, dx \right) \right] dx.$$

$$= x \sin x - \int 1 \cdot \sin x \, dx.$$

$$= x \sin x - \int \sin x \, dx.$$

$$= x \sin x - (-\cos x) + c$$

$$= x \sin x + \cos x + c$$

Imp

NOTE 1:- When there is a one function to integrate, and its integration is not known then multiply 1 and take 1 as 2^{nd} function.

Q.1 $\int \log x \, dx$.

$= \int \log x \cdot 1 \, dx$.   1st = $\log x$

2nd = 1

$= \log x \int 1 \, dx - \int \left[\left(\frac{d}{dx} \log x \right) \left(\int 1 \, dx \right) \right] dx$

$= (\log x) x - \int \frac{1}{x} \cdot x \, dx$

$= x \log x - \int dx$

$= x \log x - x + c$

formula

① $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} \left[a \cos bx + b \sin bx \right] + c$

② $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} \left[a \sin bx - b \cos bx \right] + c$

NOTE 2:-

$$\int e^x [f(x) + f'(x)]\, dx$$

$$= e^x f(x) + c$$

Ex:- $\int e^x \left[\frac{1}{x} - \frac{1}{x^2}\right] dx$

$$= \int e^x \left[\frac{1}{x} + \left(-\frac{1}{x^2}\right)\right] dx$$

$$= e^x \frac{1}{x} + c \qquad \left(\because f(x) = \frac{1}{x}, \quad f'(x) = -\frac{1}{x^2}\right)$$

—o—

Definite Integration

$$\int_a^b f(x)\,dx = g(x)+c\Big|_a^b$$

$$= \{g(b)+c\} - \{g(a)+c\}$$

$$= g(b)+c - g(a)-c$$

$$= g(b) - g(a)$$

Ex:- $\int_2^3 x^3 \cdot dx$

$$= \frac{x^4}{4}\Big|_2^3$$

$$= \frac{(3)^4}{4} - \frac{(2)^4}{4}$$

$$= \frac{65}{4}$$

SPECIAL CASE

① $\int_a^b [x]\,dx$.

$$[x] = \begin{cases} 0, & 0<x<1 \\ 1, & 1<x<2 \\ 2, & 2<x<3 \\ \cdots & \cdots \\ n-1, & n-1<x<n \end{cases}$$

Ex:- $\int_1^4 [x]\,dx$

$= \int_1^2 [x]\,dx + \int_2^3 [x]\,dx + \int_3^4 [x]\,dx$

$= \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^4 3\,dx$.

$= x\Big|_1^2 + 2x\Big|_2^3 + 3x\Big|_3^4$

$= (2-1) + (6-4) + (12-9)$

$= 1+2+3$

$= 6$

(2) $\int_{-a}^{a} |x| \, dx$.

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x > 0. \end{cases}$$

Ex:- $\int_{-3}^{3} |x| \, dx$.

$$= \int_{-3}^{0} |x| \, dx + \int_{0}^{3} |x| \, dx.$$

$$= \int_{-3}^{0} -x \, dx + \int_{0}^{3} x \, dx$$

$$= \int_{0}^{-3} x \, dx + \int_{0}^{3} x \, dx.$$

$$= \frac{x^2}{2}\Big|_{0}^{-3} + \frac{x^2}{2}\Big|_{0}^{3}$$

$$= \left\{\frac{(-3)^2}{2} - \frac{(0)^2}{2}\right\} + \left\{\frac{(3)^2}{2} - \frac{(0)^2}{2}\right\}$$

$$= \frac{9}{2} + \frac{9}{2}$$

$$= \frac{18}{2} = 9 \text{ (Ans).}$$

Properties

① $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(y)\,dy$

② $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

③ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \int_d^b f(x)\,dx$

where $a < c < d < b$

④ $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

⑤ $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & f(x) \text{ is even} \\ 0, & f(x) \text{ is odd.} \end{cases}$

NOTE: ① $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}\,dx = \frac{\pi}{4}$

② $\int_0^{\pi/2} \frac{dx}{1+\tan x} = \frac{\pi}{4}$

③ $\int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}\,dx = \frac{\pi}{4}$

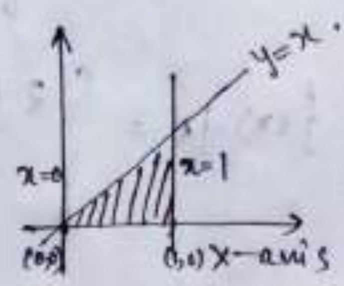
④ $\int_0^{\pi/2} \log \tan x\,dx = 0$

⑤ $\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \frac{\pi}{8}\log 2$

AREA UNDER THE CURVE

* Area under the curve w.r.t x-axis.
* Area under the curve w.r.t Y-axis.

Q.1 find the area bounded by $y=x$, x-axis, $x=0$ and $x=1$



$$\text{Area} = \int_0^1 y\, dx.$$

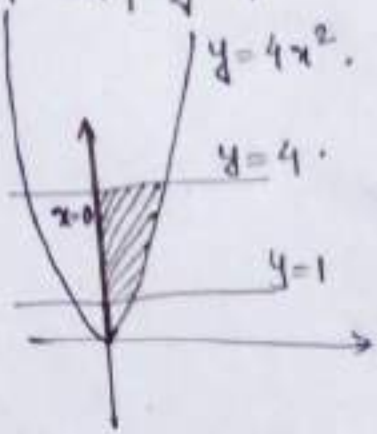
$$= \int_0^1 x\, dx$$

$$= \left.\frac{x^2}{2}\right|_0^1$$

$$= \frac{1}{2} \text{ Sq unit.}$$

Q.2 Find the area bounded by

$y = 4x^2, x = 0, y = 1$ and $y = 4$



$$\text{Area} = \int_1^4 x \, dy.$$

$$= \int_1^4 \frac{1}{2}\sqrt{y}\, dy$$

$$= \frac{1}{2}\left\{ \frac{y^{3/2}}{3/2} \Big|_1^4 \right\}$$

$$= \frac{1}{2} \times \frac{2}{3} \left\{ y^{3/2} \Big|_1^4 \right\}$$

$$= \frac{1}{3}\left[(4)^{3/2} - (1)^{3/2} \right] \text{ sq unit.}$$

$$= \frac{1}{3}(8-1) \text{ sq unit.}$$

$$= 7/3 \text{ sq unit}$$

<u>NOTE</u> :- Area bounded by the circle $x^2 + y^2 = a^2$ is $\boxed{\pi a^2}$

Ex:- Area bounded by the circle $x^2 + y^2 = 9$ is $\boxed{9\pi}$

Differential Equations

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Definition

- An equation involving
  - independent variable,
  - dependent variable and
  - derivative of dependent variable with respective to the independent variable or variables
- is known as **DIFFERENTIAL EQUATION**.

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For example:

$$\frac{dy}{dx} + 3y^2 = 9x$$

- In the above equation:
  - x = independent variable
  - y = dependent variable
  - $\frac{dy}{dx}$ = derivative of dependent variable (i.e. 'y') with respective to the independent variable or variables (i.e. 'x')

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Types of Differential Equations

- Differential Equations are of 2 types:

A. Ordinary differential equations (O.D.E)

B. Partial differential equations (P.D.E)

Video links

Ordinary differential equations (O.D.E)

- Differential equations involving derivatives w.r.t only one independent variable is called Ordinary differential equations (O.D.E)

Example:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 9x = 0$$

- Here the derivatives includes only one independent variable i.e. 'x'

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Partial differential equations (P.D.E)

- Differential equations involving derivatives w.r.t more than one independent variable is called Partial differential equations (P.D.E)

Example:

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=5u$$

Here u = f (x ,y, z), therefore

- u → dependent variable
- x, y, z → independent variables

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Order of the Differential equation

- Order of the differential equation is the highest order of the derivatives occurring in it.
- As we already know:

$\frac{dy}{dx}$ ⟹ 1st order derivative

$\frac{d^2y}{dx^2}$ ⟹ 2nd order derivative

$\frac{d^3y}{dx^3}$ ⟹ 3rd order derivative

$\frac{d^ny}{dx^n}$ ⟹ nth order derivative

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Lets see few examples:

E.g. 1:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 9x = 0$$

- Order = 2

E.g. 2:

$$\left(\frac{dy}{dx}\right) + x^2 = \frac{d^3y}{dx^3}$$

- Order = 3

Degree of the Differential equation

- Degree of the Differential equation is the highest power of the highest order derivative after the equation has been freed from radicals and fractions.

Lets see few examples:

E.g. 1:

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 = 9x$$

- Order = 3
- Degree = 1

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E.g. 2:

$$\frac{d^2y}{dx^2} = \sqrt{3 + \frac{dy}{dx}}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = 3 + \frac{dy}{dx}$$ [*squaring both sides*]

- Order = 2
- Degree = 2

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E.g. 3:

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2} = 3\left(\frac{d^2y}{dx^2}\right)$$

[squaring both sides]

$$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^{5} = \left\{3\left(\frac{d^2y}{dx^2}\right)\right\}^2$$

$$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^{5} = 9\left(\frac{d^2y}{dx^2}\right)^2$$

- Order = 2
- Degree = 2

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Solution of Differential equation

- Let us take a differential eq^n and a function

$$\frac{d^2y}{dx^2} + y = 0 \quad (1)$$

$$y = a \sin(x+b) \quad (2)$$

[*where a, b are real number*]

then

$$\Rightarrow \frac{dy}{dx} = a\cos(x+b)$$ [*differentiating eq^n* (2)]

$$\Rightarrow \frac{d^2y}{dx^2} = -a\sin(x+b)$$ [*differentiating again*]

contd..

Video links

contd..

now putting the values of y & $\frac{d^2y}{dx^2}$ in eq^n (1)

L.H.S ⇨ $\frac{d^2y}{dx^2} + y = - a \sin(x+b) + a \sin(x+b) = 0$

R.H.S ⇨ 0

L.H.S = R.H.S

- so we conclude that:

$y = a \sin(x+b)$ is solution of differential equation

$\frac{d^2y}{dx^2} + y = 0$ as it satisfies the equation.

Note:- a function is said to be solution of a differential equation if it satisfies the equation.

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Two types of solution

A. General or complete solution

B. Particular solution

Video links

General or complete solution

- A solution which contains the number of arbitrary constant equal to the order of the differential equation is called a general solution.

Example:

$y = a \sin(x+b)$ is general solution of differential equation $\frac{d^2y}{dx^2} + y = 0$

- Order of differential equation = 2
- **a**, **b** are two arbitrary constants in the solution.

Video links

Particular solution

- A particular solution of a differential equation is a solution obtained from the general solution by giving some particular values to the arbitrary constants.

Example:

$y = 2\sin(x+5)$ is particular solution of differential equation $\frac{d^2y}{dx^2} + y = 0$

Video links

Solution of Differential equation

Solution of 1st order and 1st degree equation by:

A. **Separation of variables**

B. **Solution of linear Differential equation of first order**

Video links

Separation of variables

- Consider the Differential equation

$$\frac{dy}{dx} = f(x,y) \quad (1)$$

- Equation (1) can be separable of variables

$$\Rightarrow \frac{dy}{dx} = f_1(x)\, f_2(y)$$

$$\Rightarrow \frac{dy}{f_2(y)} = f_1(x)\, dx$$

- Integrating both sides

$$\Rightarrow \int \frac{dy}{f_2(y)} = \int f_1(x)\, dx + C$$

- Which is a complete solution

Video links

Question 1

- Solve $$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

- Sol^n

[*cross-multiplying*]

$$\Rightarrow \frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

[*integrating both sides*]

$$\Rightarrow \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$$\Rightarrow \tan^{-1} y = \tan^{-1} x + C$$ answer

Video links

Question 2

- Solve

$$e^x \tan y \, dx + (1+e^x)\sec^2 y \, dy = 0$$

- Sol[n]

$\Rightarrow$ $e^x \tan y \, dx + (1+e^x)\sec^2 y \, dy = 0$

$\Rightarrow$ $(1+e^x)\sec^2 y \, dy = -e^x \tan y \, dx$

$\Rightarrow$ $\dfrac{\sec^2 y \, dy}{\tan y} = \dfrac{-e^x \, dx}{(1+e^x)}$

[*integrating both sides*]

$\Rightarrow$ $\displaystyle\int \frac{\sec^2 y \, dy}{\tan y} = \int \frac{-e^x \, dx}{(1+e^x)}$ —— (1)

I_1 $\quad$ I_2

contd..

Video links

contd..

- For I_1

Let $\tan y = u$

$\Rightarrow \sec^2 y = \frac{du}{dy}$

$\Rightarrow \sec^2 y \, dy = du$

$\Rightarrow \int \frac{\sec^2 y \, dy}{\tan y} = \int \frac{du}{u}$

$\Rightarrow = \log u$

$\Rightarrow = \log \tan y$

- For I_2

Let $(1 + e^x) = v$

$\Rightarrow e^x = \frac{dv}{dx}$

$\Rightarrow e^x \, dx = dv$

$\Rightarrow -\int \frac{e^x \, dx}{(1 + e^x)} = -\int \frac{dv}{v}$

$\Rightarrow = - \log v$

$\Rightarrow = - \log (1 + e^x)$

Eg^n (1) becomes:

$\Rightarrow \log \tan y = - \log (1 + e^x) + C$ answer

Video links

Solution of linear Differential equation of first order

- A differential equation in which the dependent variable and all its derivatives occur in the 1st degree only and are not multiplied together is called a Linear Differential equation.
- **Standard form** of linear differential equation (1st order)

$$\frac{dy}{dx} + Py = Q$$

- where P and Q may be constant or only a function of x.
- coefficient of $\frac{dy}{dx}$ is always unity.

contd..

Video links

contd..

method of solution

- Step 1
  - Find I.F (Integrating factor)

$$\Rightarrow \quad e^{\int p\, dx}$$

- Step 2
  - Then the complete solution is given by

$$y \times I.F = \int \{Q \times (I.F)\}\, dx + C$$

Video links

Question 1

- Solve $$\frac{dy}{dx} + y\,\tan x = \sec x$$

- Sol^n It is in its standard form

$$\Rightarrow \frac{dy}{dx} + Py = Q$$

[$P = \tan x$]
[$Q = \sec x$]

I.F (Integrating factor) $\rightarrow e^{\int p\,dx}$

$$\Rightarrow e^{\int \tan x\,dx}$$

$$\Rightarrow e^{\log|\sec x|}$$

$$\Rightarrow \sec x$$

contd..

Video links

contd..

complete solution is given by:

$$y \times I.F = \int\{Q \times (I.F)\}\,dx + C$$

$$\Rightarrow \quad y \times \sec\mathbf{x} = \int\{\sec\mathbf{x} \times \sec\mathbf{x}\}\,dx + C$$

$$\Rightarrow \quad y \sec\mathbf{x} = \int\{\sec^2\mathbf{x}\}\,dx + C$$

$$\Rightarrow \quad y \sec\mathbf{x} = \tan\mathbf{x} + C$$

answer

Video links

Question 2

- Solve $$x\frac{dy}{dx} + 2y = 4x^2$$

- Sol^n

it is not in its standard form

$$\Rightarrow \frac{dy}{dx} + \frac{2y}{x} = 4x$$

[*divide by 'x' on both sides*]

now it is in the standard form

$[\, P = \frac{2}{x} \,]$

$[\, Q = 4x \,]$

I.F $\Rightarrow e^{\int p\,dx}$

$$\Rightarrow e^{\int \frac{2}{x}dx} \Rightarrow e^{2\int \frac{1}{x}dx}$$

$$\Rightarrow e^{2\log x} \Rightarrow e^{\log x^2}$$

$$\Rightarrow x^2$$

contd..

Video links

contd..

complete solution is given by:

$$y \times I.F = \int\{Q \times (I.F)\}\, dx + C$$

$\Rightarrow$ $y\, x^2 = \int(4x \,.\, x^2)\, dx + C$

$\Rightarrow$ $y\, x^2 = \int(4x^3)\, dx + C$

$\Rightarrow$ $y\, x^2 = \dfrac{4x^4}{4} + C$

$\Rightarrow$ $y\, x^2 = x^4 + C$ answer

Video links

Question 3

- Solve $$(1+x^2)\frac{dy}{dx}+2xy - x^3 = 0$$

- Sol^n

it is not in its standard form

$$\Rightarrow \frac{dy}{dx}+\frac{2x\,y}{1+x^2}-\frac{x^3}{1+x^2} = 0$$ [*divide by '1+x²' on both sides*]

$$\Rightarrow \frac{dy}{dx}+\frac{2x\,y}{1+x^2} = \frac{x^3}{1+x^2}$$

[$P = \frac{2x}{1+x^2}$]

[$Q = \frac{x^3}{1+x^2}$]

now it is in the standard form

contd..

Video links

contd..

I.F $\Rightarrow e^{\int p\,dx}$

$\Rightarrow e^{\int \frac{2x}{1+x^2}\,dx}$

$\Rightarrow e^{\int \frac{1}{t}\,dt}$

$\Rightarrow e^{\log t}$

$\Rightarrow t = 1+x^2$

[*let* $1+x^2 = t$]

[$2x = \frac{dt}{dx}$]

[$2x\,dx = dt$]

Video links

contd..

complete solution is given by:

$$y \times I.F = \int\{Q \times (I.F)\}\, dx + C$$

$$\Rightarrow\quad y\,(1+x^2) = \int\left(\frac{x^3}{1+x^2}\right)(1+x^2)\, dx + C$$

$$\Rightarrow\quad y\,(1+x^2) = \int x^3\, dx + C$$

$$\Rightarrow\quad y\,(1+x^2) = \frac{x^4}{4} + C$$

answer

Video links

Vectors

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Scalars and Vectors

A **scalar quantity** is a quantity that has only **magnitude**.

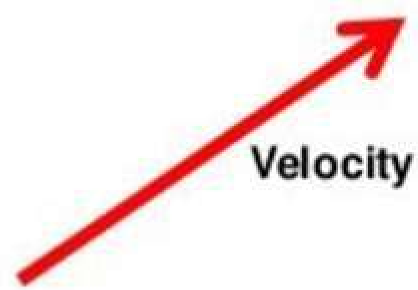
A **vector quantity** is a quantity that has both a **magnitude** and a **direction**.

Scalar quantities

Length, Area, Volume,
Speed,
Mass, Density
Temperature, Pressure
Energy, Entropy
Work, Power

Vector quantities

Displacement, Direction,
Velocity, Acceleration,
Momentum, Force,
Electric field, Magnetic field



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scalar

- only magnitude (size)
- 3.044, −7 and 2½

Example:

- Distance = 3 km
- Speed = 9 km/h
  (kilometers per hour)

vector

- magnitude and direction

- Displacement = 3 km
  Southeast
- Velocity = 9 km/h
  Westwards

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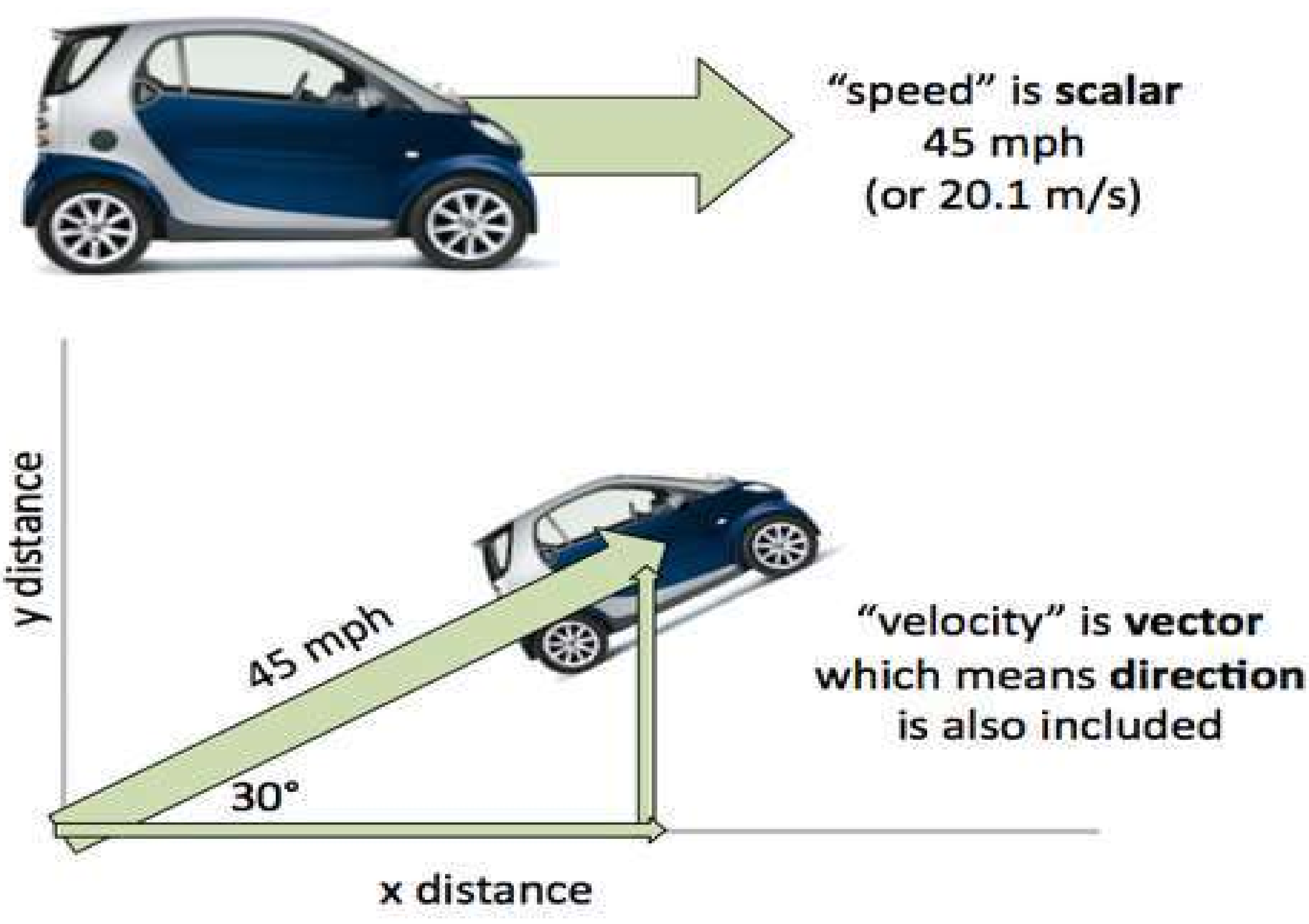
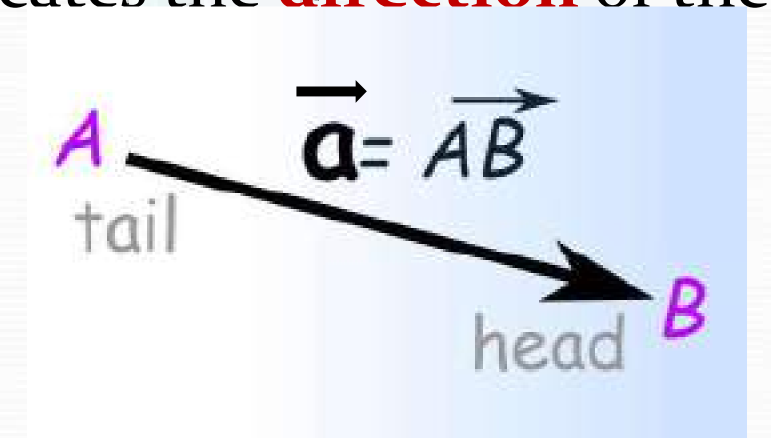
Distance is a scalar quantity, whereas displacement is a vector quantity.

Scalar and Vector Quantities



Vector - Notation/ Denoted as

- It is denoted as 'vector $\overrightarrow{AB}$' or 'vector $\vec{a}$'.
- point A from where the vector starts is called its initial point
- point B where it ends is called its terminal point.
- The distance between initial and terminal points of a vector is called the **magnitude** (or length) of the vector, denoted as $|\overrightarrow{AB}|$, or $|\vec{a}|$, or a.
- The arrow indicates the **direction** of the vector.



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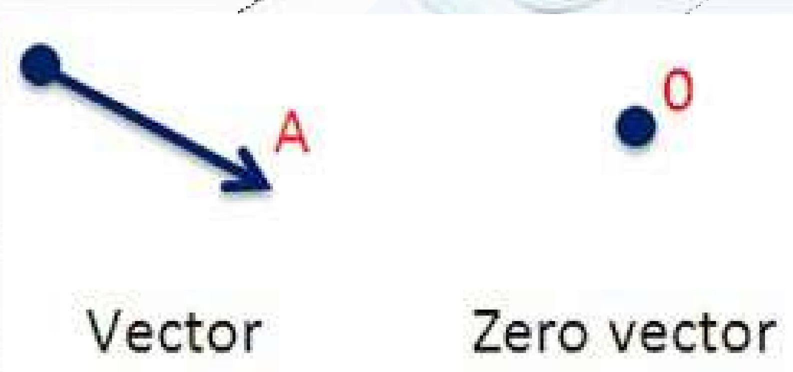
Types of vector

- zero or null vector
- unit vector
- negative of a vector
- co-initial vectors
- co-terminus vectors
- equal vectors
- collinear or parallel vectors

Video links

zero or null vector

- initial and terminal points coincident
- denoted by ➡ $\vec{0}$
- Magnitude ➡ 0 (zero)



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unit vector

- Magnitude ➡ **1** (unit magnitude, A= 1)
- denoted as ➡ $\hat{a}$
- purpose ➡ specify a **direction** in space

VECTOR A ←

$$\vec{A} = A\,\hat{A}$$

A = magnitude of $\vec{A}$

$\hat{A}$ = unit vector along $\vec{A}$

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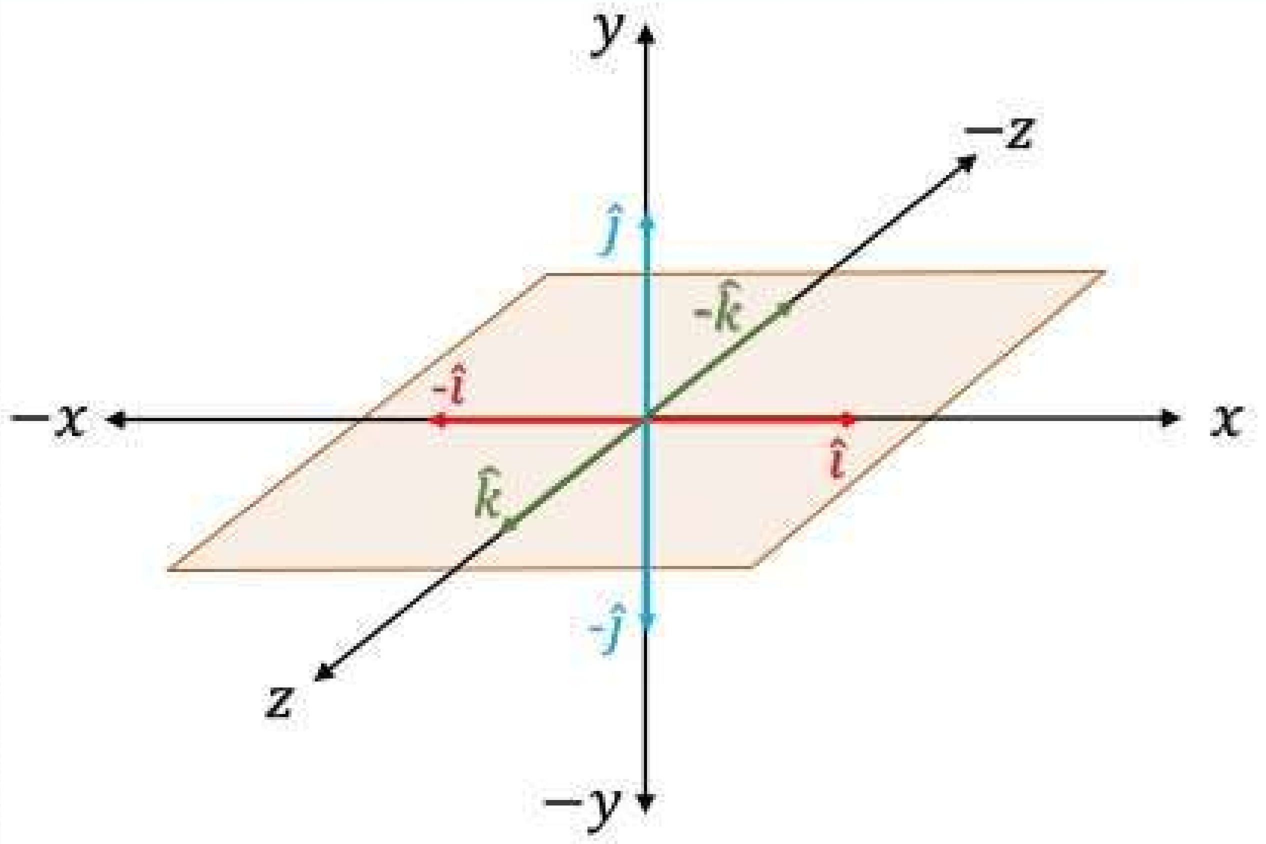
Cartesian unit vectors



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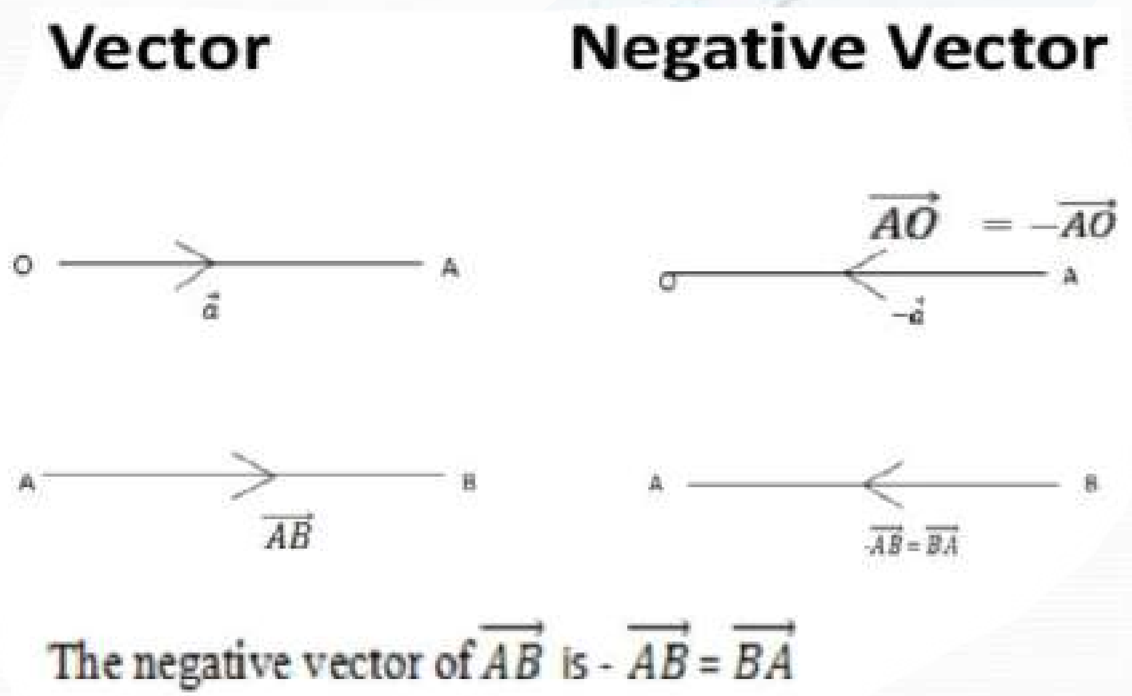
negative of a vector

- Vector of same magnitude
- but opposite direction



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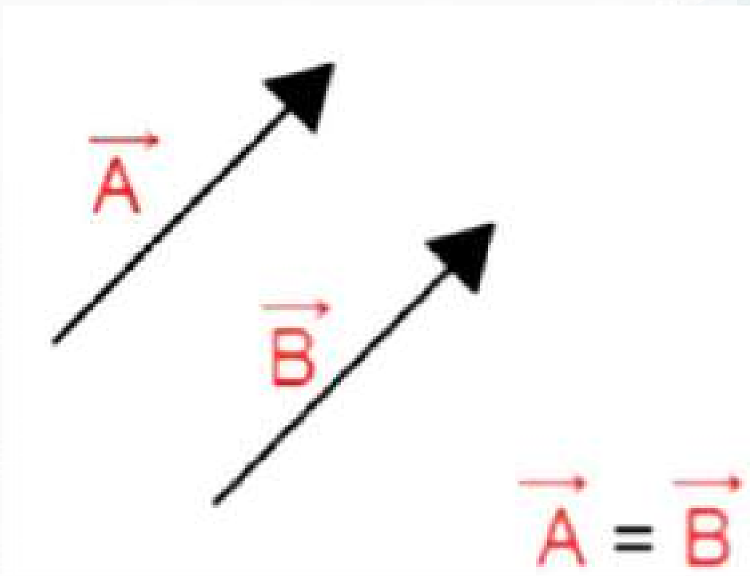
equal vectors

- same magnitude (size) as well as direction



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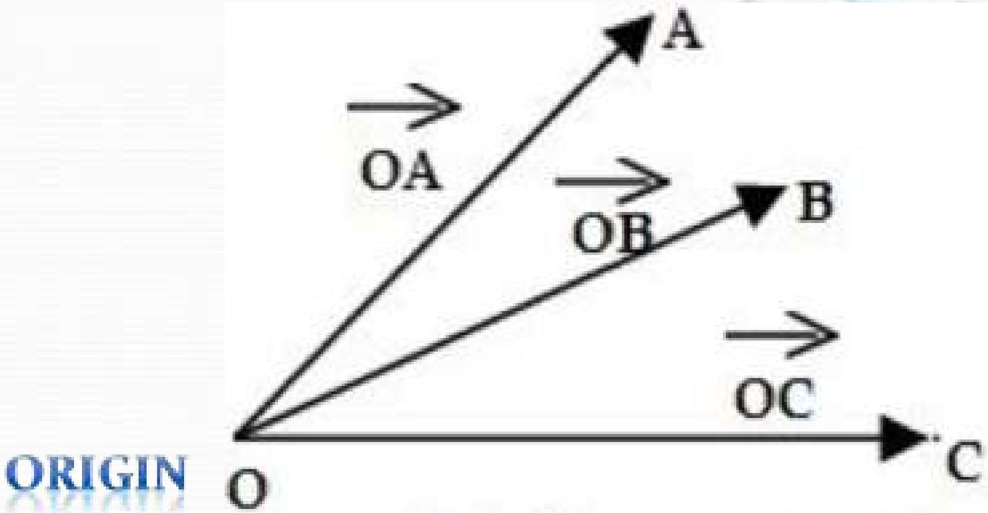
co-initial vectors

- same starting point



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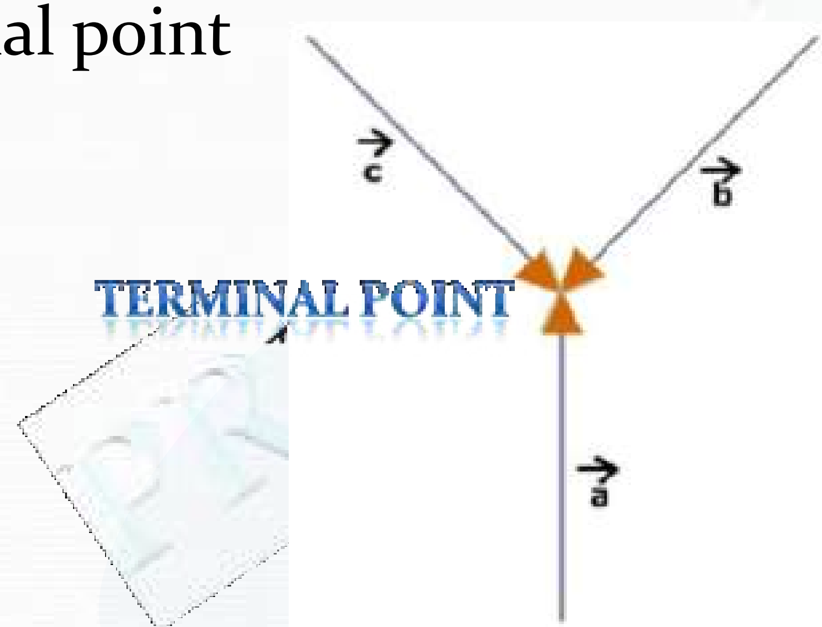
co-terminus vectors

- same terminal point



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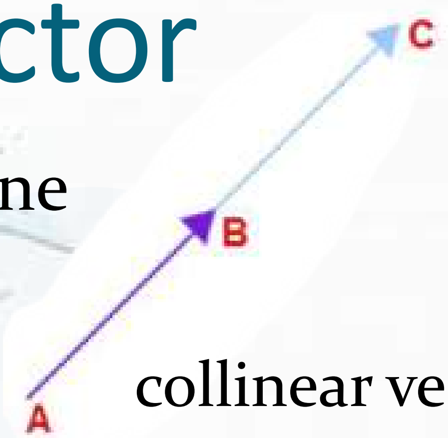
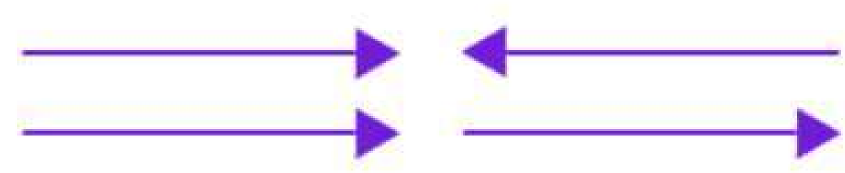
collinear or parallel vector

- **collinear vectors** ➡ lying on one line



collinear vector

- **parallel vectors** ➡ lying parallel to each other

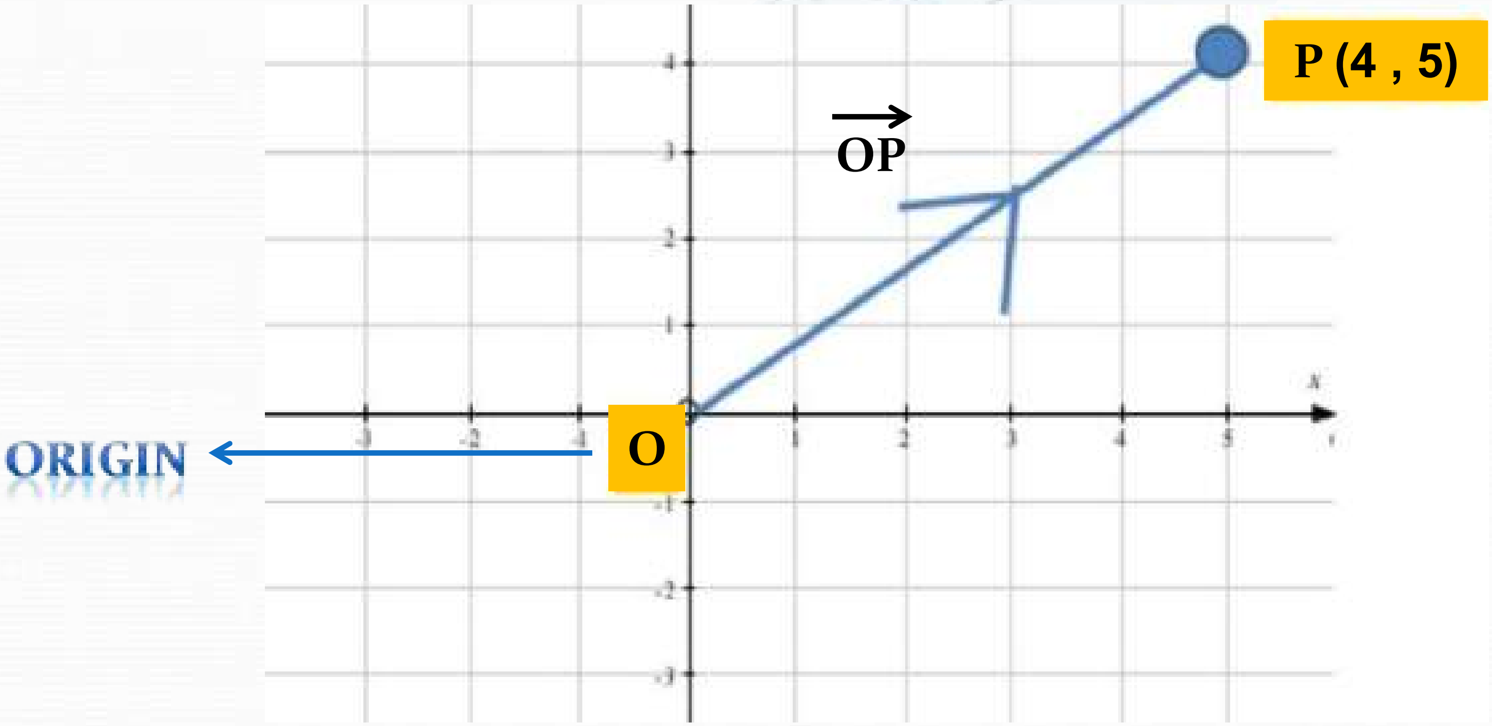


parallel vector

Video links

position vector

- Vector having initial point is at origin. Here $\overrightarrow{OP}$ is the position vector of point 'P'.



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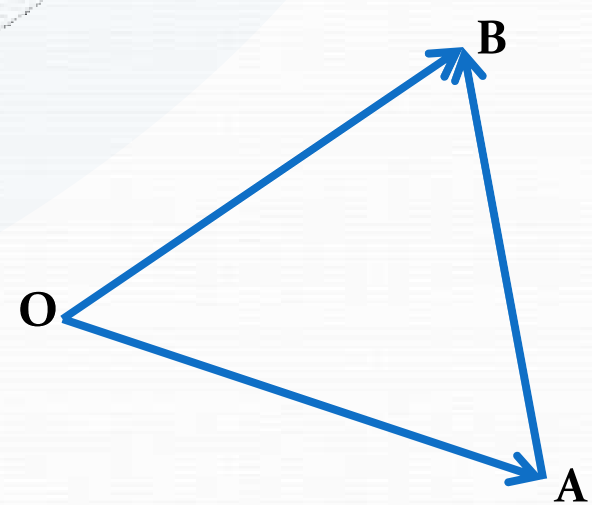
Representation of vectors in terms of the position vectors

- Let A and B be two given points.
- Then $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the position vectors of A and B
- Then $\overrightarrow{AB}$ can be represented as:

➡ $\overrightarrow{AB}$ = p.v. of B – p.v. of A

➡ $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$



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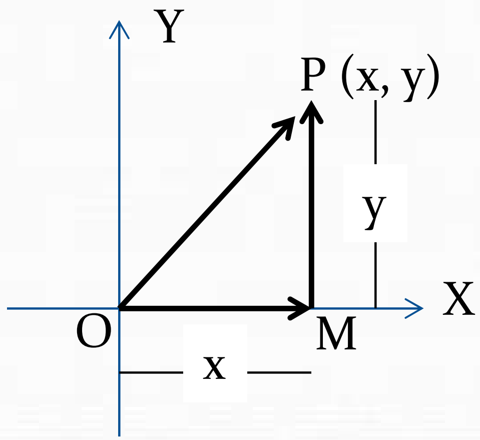
Components of a vector in two dimensions



Let $\hat{i}$ and $\hat{j}$ be the unit vectors along x-axis and y-axis

Then $\overrightarrow{OM} = x\hat{i}$

$\overrightarrow{MP} = y\hat{j}$

Then $\overrightarrow{OP} = x\,\hat{i} + y\,\hat{j}$ *[by Triangle law of addition]*

$|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$

as in Δ OPM

$(OP)^2 = (OM)^2 + (PM)^2$

$\Rightarrow (OP)^2 = x^2 + y^2$

$\Rightarrow OP = \sqrt{x^2 + y^2}$

Video links

Components of a vector in three dimensions

Let P (x, y, z) be a point in 3D

Here $\hat{i}$, $\hat{j}$ & $\hat{k}$ are unit vectors along X-axis, Y-axis & Z-axis respectively

Then

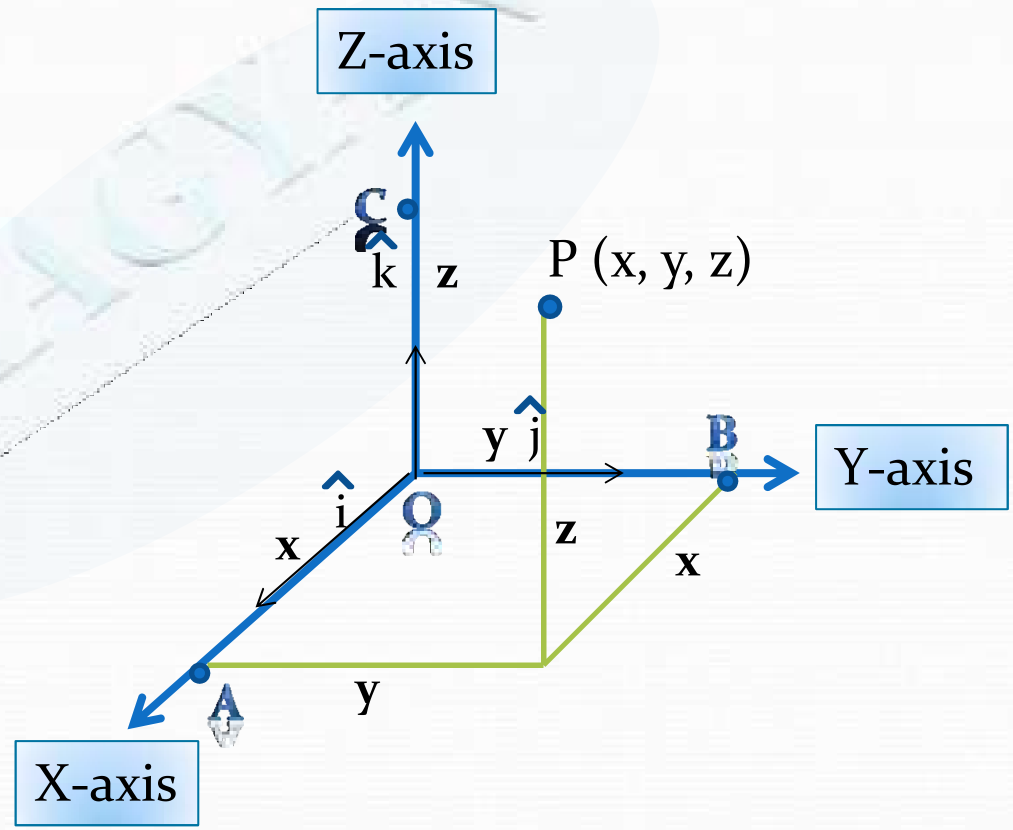
$\vec{OA} = x\,\hat{i}$

$\vec{OB} = y\,\hat{j}$

$\vec{OC} = z\,\hat{k}$

So $OP = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

and $|OP| = \sqrt{x^2 + y^2 + z^2}$



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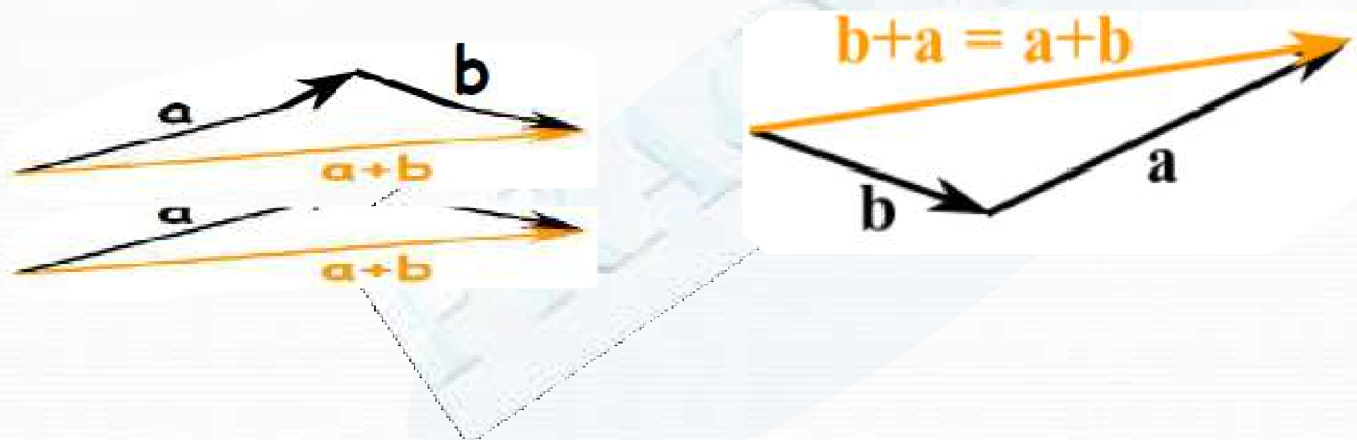
Operations on vectors

- Addition of two vectors
  - Triangle law of addition
  - Parallelogram law of addition
- Subtraction of two vectors
- Multiplication
  - of a vector with a scalar
  - of two vectors by Dot product
  - of two vectors Cross product

Video links

Adding Vectors by triangle law of addition

- We can add two vectors by joining them **head-to-tail**



triangle law of vector addition – states that if two vectors represented by 2 sides of the triangle then their sum is represented by the third side of the triangle but in the reverse order.

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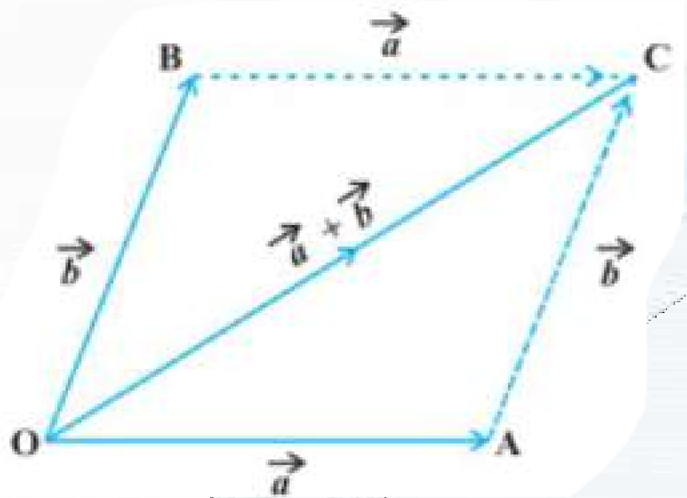
Adding Vectors by parallelogram law of vectors

- We can also add two vectors having a **same origin**

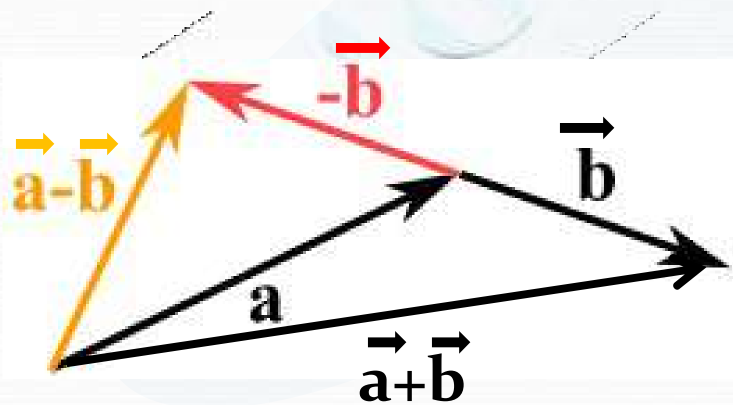


parallelogram law of vector addition – states that if 2 vectors $\vec{a}$ & $\vec{b}$ are represented by 2 adjacent sides of a parallelogram, then their sum $\vec{a} + \vec{b}$ is represented by the diagonal of the paralleogram through their initial point.

Video links

Subtracting vectors

- Let $\vec{a}$ and $\vec{b}$ be two vectors, reverse the direction of the vector $\vec{b}$ then add as usual:



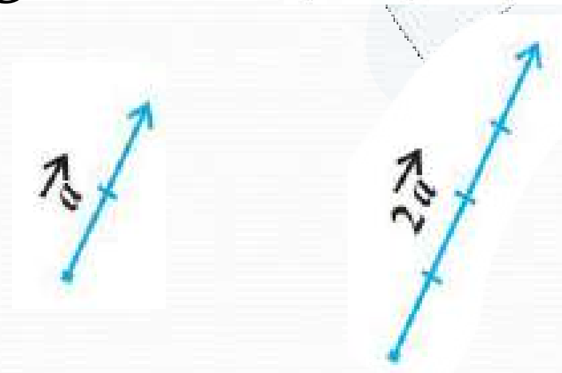
Video links

Multiplying a Vector by a Scalar

- product of the vector $\vec{a}$ by the scalar $\lambda = \lambda\vec{a}$
- magnitude $\Rightarrow |\lambda\vec{a}| = |\lambda||\vec{a}|$

Example: $\vec{a}$ x 2 = $2\vec{a}$

magnitude = $|2\vec{a}| = |2||\vec{a}|$ = 2a



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Addition of two vectors in components

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then $\overrightarrow{a + b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$\Rightarrow (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

Subtraction of two vectors in components

Then $\overrightarrow{a - b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) - (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$\Rightarrow (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$

Video links

Multiplication of a vector with scalar

Let λ be a scalar

$$a = a_1 i + a_2 j + a_3 k$$

Then $\lambda \vec{a} = \lambda (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k})$

$$\lambda a_1 \hat{i} + \lambda a_2 \hat{j} + \lambda a_3 \hat{k}$$

Video links

Multiplication of 2 vectors

- By using Scalar/ Dot product
- By using Vector/ Cross product

Video links

Scalar or Dot Product

- Let $\vec{a}$ & $\vec{b}$ be two vectors.
- Then dot product of them is denoted by $\vec{\mathbf{a}} \,.\, \vec{\mathbf{b}}$
- and defined as:

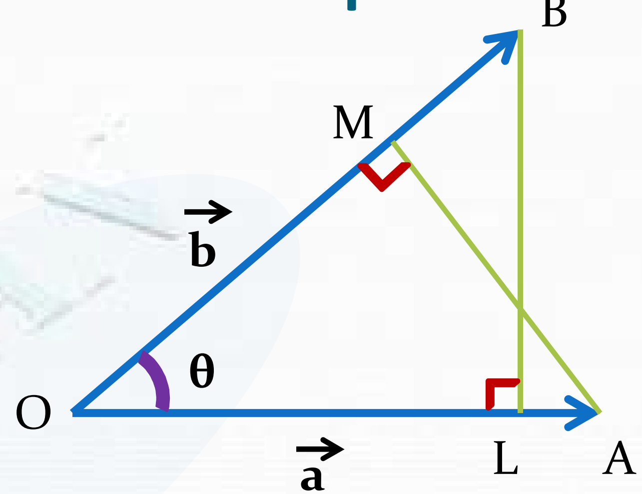
$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}| \times |\vec{\mathbf{b}}| \times \cos(\theta)$$

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \mathbf{a} \times \mathbf{b} \times \cos(\theta)$$

or $$\cos(\theta) = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}| \; |\vec{\mathbf{b}}|}$$

Video links

Geometrical representation of Dot product



Here in the given figure
θ is the angle between the vectors $\vec{a}$ & $\vec{b}$

Consider the right angled triangle ΔOBL then

$$\cos\theta = \frac{b}{h} = \frac{OL}{OB} = \frac{OL}{|\vec{b}|}$$

$$|\vec{b}|\cos\theta = OL$$

and OL is known as projection of $\vec{b}$ on $\vec{a}$

as we know $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,OL$$

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = OL$$

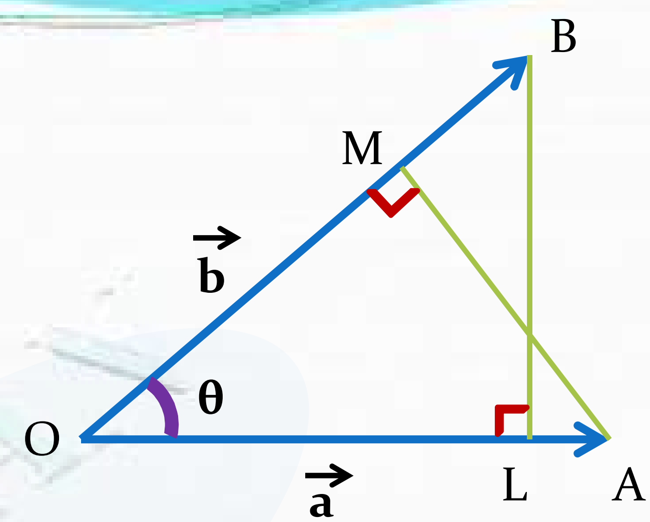
So scalar projection of $\vec{b}$ on $\vec{a}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

Continued..

Video links

Continued..



Again consider the right angled triangle ΔOAM then

$$\cos\theta = \frac{b}{h} = \frac{OM}{OA} = \frac{OM}{|\vec{a}|}$$

$$|\vec{a}| \cos\theta = OM$$

and OM is projection of $\vec{a}$ on $\vec{b}$

as we know $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

$$\vec{a} \cdot \vec{b} = |\vec{b}|\, OM$$

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = OM$$

So scalar projection of $\vec{a}$ on $\vec{b}$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

Video links

Dot product in terms of components

Let
$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

We have

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$$
or $$\hat{j}\cdot\hat{i} = \hat{k}\cdot\hat{j} = \hat{i}\cdot\hat{k} = 0$$ — 1

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$$ — 2

Then
$$\vec{a}\cdot\vec{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)\cdot\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$$

1
$$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

2
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} \Rightarrow \cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

Video links

Continued..

Continued..

3. If $\vec{a}$ is perpendicular to $\vec{b}$

Then $\theta = 90^o$ ➡ $\cos\theta = \cos 90^o = 0$

➡ $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$

➡ $0 = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$

➡ $0 = \vec{a}\cdot\vec{b}$

So $\vec{a} \perp \vec{b}$ ➡ $\vec{a}\cdot\vec{b} = 0$

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4 If $\vec{a}$ & $\vec{b}$ are parallel to each other

$$\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

5 $$\vec{a} \cdot \vec{a} = |\vec{a}|\,|\vec{a}| \cos 0$$

$$\Rightarrow |\vec{a}|^2 \cdot 1$$

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

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Vector or Cross Product

- The Vector Product of two vectors is denoted by $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ and defined as:

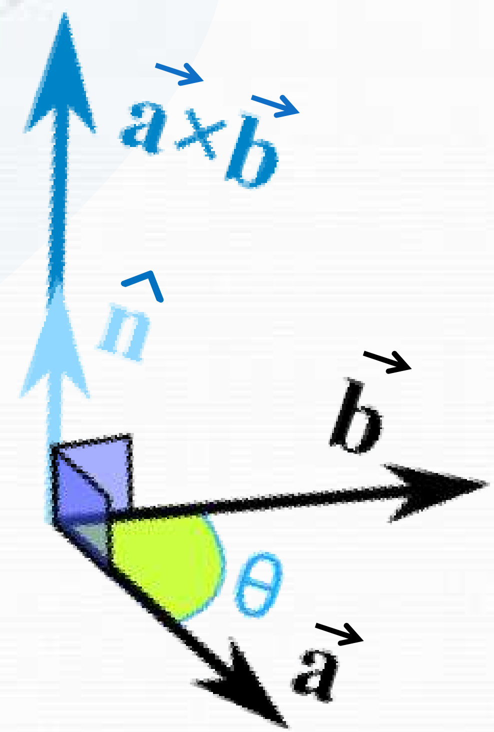
$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = |\vec{\mathbf{a}}|\ |\vec{\mathbf{b}}| \sin\theta \ .\ \hat{\mathbf{n}}$$

where:

$|\vec{a}|$ & $|\vec{b}|$ = magnitude

θ = angle between $\vec{a}$ & $\vec{b}$

$\hat{n}$ = unit vector perpendicular to both $\vec{a}$ & $\vec{b}$



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Note:- 1

we have $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta \,.\, \hat{n}$

$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta \,.\, |\hat{n}|$

$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$ $\quad \{ |\hat{n}| = 1 \}$

$$\Rightarrow \frac{\vec{a} \times \vec{b}}{|\vec{a}|\,|\vec{b}|} = \sin\theta$$

Note:- 2

we have $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta \,.\, \hat{n}$

$$\Rightarrow \vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|} \hat{n} \quad \{\text{putting formula of } \sin\theta\}$$

$\Rightarrow \vec{a} \times \vec{b} = |\vec{a} \times \vec{b}|\,\hat{n}$ $\Rightarrow$ $$\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \hat{n}$$

So a unit vector $\hat{n}$ perpendicular to both $\vec{a}$ & $\vec{b}$ is given by

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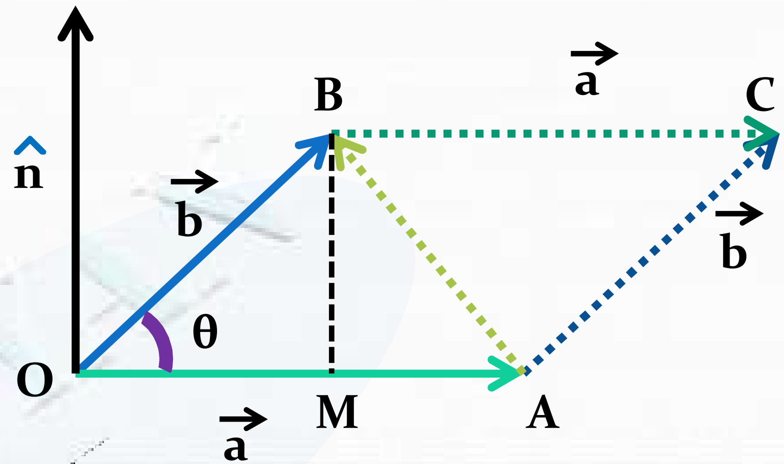
Geometrical representation of vector product

$\vec{OA} = \vec{a}$

$\vec{OB} = \vec{b}$



$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta \;\hat{n}$

⇒ $\vec{a} \times \vec{b} = |\vec{a}|\,(|\vec{b}| \sin\theta)\;\hat{n}$

$\sin\theta = \dfrac{p}{h} = \dfrac{BM}{OB} = \dfrac{BM}{|\vec{b}|}$

$|\vec{b}| \sin\theta = BM$

⇒ $\vec{a} \times \vec{b} = |\vec{a}|\; BM \;\hat{n}$

⇒ $|\vec{a} \times \vec{b}| = |\vec{a}|\,|BM|$

$=$ area of a parallelogram with sides $\vec{a}$ & $\vec{b}$

area of a parallelogram
= base × height

Then it is concluded that:

Area of Δ ABC = ½ $|\vec{a} \times \vec{b}|$

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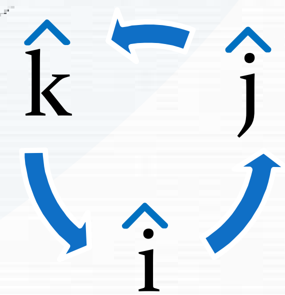
Vector product in terms of components

Let

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

And from a right handed system of mutually perpendicular vector
We have:

$$\hat{i}\times\hat{j} = \hat{k} \quad \text{or} \quad \hat{j}\times\hat{i} = -\hat{k}$$
$$\hat{j}\times\hat{k} = \hat{i} \qquad \hat{k}\times\hat{j} = -\hat{i}$$
$$\hat{k}\times\hat{i} = \hat{j} \qquad \hat{i}\times\hat{k} = -\hat{j}$$



And

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$$

So

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

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