



LECTURE NOTE
ON
ENGG. MATHEMATICS-I

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INTRODUCTION TO MATRIX:

Equation :-

It is a statement of two Mathematical expression indicated by an equal (=) sign.

$$\text{for example } \rightarrow \underline{3x^2 - 5} = \underline{4x + 3}$$

where $3x^2 - 5$ and $4x + 3$ are two Mathematical expressions.

Types :-

There are different types of equations, such as

- ① Linear Equation (Degree 1)
- ② Quadratic Equation (Degree 2)
- ③ Cubic Equation (Degree 3)
- ④ Quartic Equation (Degree 4)
- ⑤ Quintic Equation (Degree 5)

and so... on...

LINEAR EQUATIONS

Linear Equations are the equations of degree '1'.

Such as -

(i) $4x+3=5x+9$ (Linear equation of one Variable)

(ii) $3x+4y=9$ (Linear equation of two Variables)

(iii) $2x-3y+4z=0$ (Linear equation of three Variables)

Solution of linear Equations

(1) Solution of linear eqⁿ of one Variable.

Q:- Solve $9x+4=11x-5$

Solⁿ:- Given $9x+4=11x-5$

$$\Rightarrow 9x-11x = -5-4$$

$$\Rightarrow -2x = -9$$

$$\Rightarrow x = \frac{-9}{-2} = \frac{9}{2}$$

(2) Solution of linear eq^s of two Variables.

Q:- Solve $x+y=3$
 $x+3y=5$

Now this type of linear system of equations can be solved by different Methods, like

- ① Substitution Method.
- ② Elimination Method.
- ③ cross Multiplication Method.
- ④ Graphical Method.

① Substitution Method:

Solⁿ:- Given $x+y=3$ — (1)
 $x+3y=5$ — (2)

From eqⁿ ① we get $x=3-y$

putting $x=3-y$ in eqⁿ (2)

eqⁿ ② becomes

$$\Rightarrow x+3y=5$$

$$\Rightarrow 3+2y=5$$

$$\Rightarrow 2y=2$$

$$\Rightarrow \boxed{y=1}$$

putting $y=1$ in eqⁿ ①

$$\Rightarrow x+1=3$$

$$\Rightarrow \boxed{x=2}$$

① Elimination Method:-

Solⁿ :- Given $x+y=3$ — ①
 $x+3y=5$ — ②

As already the coefficients of variable 'x' are equal so, subtract eqⁿ ② from eqⁿ ①

i.e.

$$\begin{array}{r} x+y=3 \\ +x+3y=5 \\ \hline (-) \quad (-) \quad (+) \\ \hline -2y=-2 \\ \Rightarrow \boxed{y=1} \end{array}$$

putting the value of 'y' in eqⁿ ①

we get $\boxed{x=2}$

③ Cross Multiplication Method :-

Solⁿ :- Given $x+y=3$
 $x+3y=5$

The above equations can be rewritten as

$$x+y-3=0 \quad [a_1=1, b_1=1, c_1=-3]$$

$$x+3y-5=0 \quad [a_2=1, b_2=3, c_2=-5]$$

By Method of cross Multiplication,

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

Substituting the values in the above equation

$$\Rightarrow \frac{x}{1(-5) - 3(-3)} = \frac{y}{(-3)(1) - (1)(1)} = \frac{1}{1(3) - 1(1)}$$

$$\Rightarrow \frac{x}{-5+9} = \frac{y}{-3+5} = \frac{1}{3-1}$$

$$\Rightarrow \frac{x}{4} = \frac{y}{2} = \frac{1}{2}$$

consider $\frac{x}{4} = \frac{1}{2} \Rightarrow x = \frac{4}{2} \Rightarrow \boxed{x=2}$

Again consider $\frac{y}{2} = \frac{1}{2} \Rightarrow y = \frac{2}{2} \Rightarrow \boxed{y=1}$

Graphical Method :-

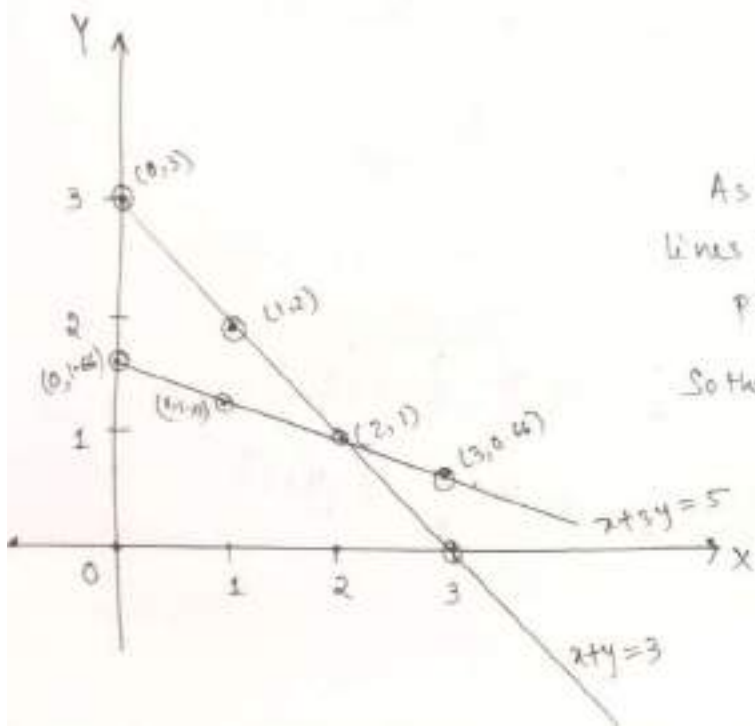
Given eq^s are $x+y=3$
 $x+3y=5$

Now for $x+y=3$

| | | | | |
|---|---|---|---|---|
| x | 0 | 1 | 2 | 3 |
| y | 3 | 2 | 1 | 0 |

for $x+3y=5$

| | | | | |
|---|------|------|---|------|
| x | 0 | 1 | 2 | 3 |
| y | 1.66 | 1.33 | 1 | 0.66 |



As both the lines intersect at the pt (2,1)

So the solⁿ is

$$\boxed{x=2}$$

$$\boxed{y=1}$$

* There is another Method for solving linear system of equations known as Matrix Method.

Application of Matrices :-

Basically Matrices is an essential Mathematical tool which helps

- (i) To represent large system of linear equations
- (ii) To find the solution of such equations

MATRIX :-

Definition :-

Matrix means arrangement of numbers in some rows and columns in a rectangular shape.
(horizontal lines) (vertical lines)

for example :-

$$A = \begin{matrix} & \begin{matrix} \uparrow c_1 & \uparrow c_2 & \uparrow c_3 & \uparrow c_4 \end{matrix} \\ \begin{matrix} \left[\begin{array}{cccc} 4 & 9 & 6 & 5 \\ -5 & 3 & 0 & 2 \\ 7 & 9 & -6 & 8 \end{array} \right] & \begin{matrix} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{matrix} \end{matrix}$$

- * Matrices are always denoted by capital letters
- * Numbers in the Matrix, known as elements of Matrix.

Order :-

The number of rows and columns that a Matrix has is called its order or dimension.

Basically rows are listed first and columns second. And is written in the form.

$M \times N$, means M rows and N columns.

For example :-

$$A = \begin{bmatrix} 3 & 4 & 9 \\ 0 & -6 & 5 \end{bmatrix}$$

Here the above Matrix has 2 rows and 3 columns.
So the order of $A = 2 \times 3$

General form of a Matrix of order $M \times N$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & \dots & a_{mn} \end{bmatrix}_{M \times N}$$

Here the elements are written in the form a_{ij} where i - rows
 j - columns.

So a_{32} means element in the 3rd row and 2nd column.

for example -

Q.1 write down a Matrix of order 4×3 .

Solⁿ

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}_{4 \times 3}$$

Q.2 write down a Matrix of order 2×3 st. $a_{ij} = i+j$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2 \times 3}$$

As $a_{ij} = i+j$

$$\text{So, } a_{11} = 1+1 = 2$$

$$a_{21} = 2+1 = 3$$

$$a_{12} = 1+2 = 3$$

$$a_{22} = 2+2 = 4$$

$$a_{13} = 1+3 = 4$$

$$a_{23} = 2+3 = 5$$

$$\text{So } A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}_{2 \times 3}$$

TYPES OF MATRIX :

① Row Matrix :-

A Matrix of order $1 \times n$ (Matrix having one row) is called a row Matrix.

for example

$$A = [1 \ 4 \ 0 \ 9]_{1 \times 4}$$

$$B = [4 \ -5 \ 6]_{1 \times 3}$$

$$C = [-7]_{1 \times 1}$$

② column Matrix :-

A Matrix of order $m \times 1$ (Matrix having one column) is called column Matrix.

for example

$$A = \begin{bmatrix} 4 \\ 9 \\ -5 \end{bmatrix}_{3 \times 1}$$

$$B = \begin{bmatrix} 4 \\ 0 \\ 9 \\ 1 \\ 6 \end{bmatrix}_{5 \times 1}$$

$$C = [9]_{1 \times 1}$$

③ Square Matrix :-

If the number of rows is equal to the number of columns, then the Matrix is called Square Matrix.

for example :-

$$A = \begin{bmatrix} 4 & 0 & -5 \\ 2 & 3 & 4 \\ 7 & 2 & 1 \end{bmatrix}_{3 \times 3} \quad (\text{order } 3)$$

$$B = \begin{bmatrix} 4 & 2 \\ 9 & 8 \end{bmatrix}_{2 \times 2} \quad (\text{order } 2)$$

$$A = [5]_{1 \times 1} \quad (\text{order } 1)$$

④ Diagonal Matrix :-

A Square Matrix is said to be a diagonal Matrix if its non-diagonal elements are zero.

for example :-

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3 \times 3}$$

⑤ Scalar Matrix :-

A diagonal Matrix is said to be scalar Matrix if all the diagonal elements are equal.

for example :-

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3 \times 3}$$

⑥ Identity / Unit Matrix :-

A square Matrix in which all the diagonal elements are '1' and rest of the elements are zero.

for example :-

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$$

$$I_1 = [1]_{1 \times 1}$$

① Zero/Null Matrix :-

A Matrix is said to be Null Matrix if all the elements are zero.

for example :-

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2}$$

② Upper-Triangular Matrix :-

A square Matrix is called a upper triangular Matrix if all the elements below the main diagonal are zero.

for example :-

$$A = \begin{bmatrix} 4 & 9 & 0 \\ 0 & 5 & -1 \\ 0 & 0 & 8 \end{bmatrix}_{3 \times 3}$$

③ Lower-Triangular Matrix :-

A square Matrix is called a lower triangular Matrix if all the elements above the main diagonal are zero.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 9 & 6 & 0 \\ 5 & 2 & 1 \end{bmatrix}$$

Equality of two Matrices

Two Matrices ^{A x B} are said to be equal

iff (i) order of A = order of B

(ii) Each element of A is equal to the corresponding element of B

for example :-

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Here (i) order of A = order of B = 2x2

and if $a=x, b=y, c=z$ & $d=w$

then $A=B$.

Transpose of a Matrix

Let 'A' be a Matrix, then its transpose is obtained by converting rows into columns or columns into rows and denoted as A^T .

for example :- $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -2 & 1 \end{bmatrix}_{2 \times 3}$

$$\text{Then } A^T = \begin{bmatrix} 2 & 3 \\ 3 & -2 \\ 4 & 1 \end{bmatrix}$$

Operations on Matrix :-

(i) Addition of two Matrix :-

We can add two Matrices if their order is same

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

$$\text{Then } A+B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

$$= \begin{bmatrix} a+x & b+y \\ c+z & d+w \end{bmatrix}$$

Note :- when we add two matrix A and B, we add each element of A with the corresponding element of B.

(ii) Subtraction :-

Similarly we can subtract two matrices if their order is same.

for example :- $A-B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

$$= \begin{bmatrix} a-x & b-y \\ c-z & d-w \end{bmatrix}$$

(iii) Multiplication :-

Case I :- Multiplication of a Matrix by a number.

Let A be a Matrix, 'k' be a number.

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{Then } kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

i.e. when a number 'k' is multiplied to Matrix, actually it is multiplied with each and every element of the matrix.

Case II :- Multiplication of two Square Matrices.

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2 \times 2} \quad \text{and } B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}_{2 \times 2} \text{ be}$$

two square matrices of order 2

$$\text{Then } AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

by Row-column Method.

$$= \begin{bmatrix} ax+by & ay+bw \\ cx+dz & cy+dw \end{bmatrix}$$

Q.1 find AB and BA

where $A = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$

Solution :- $AB = \begin{matrix} \xrightarrow{\text{row}} \\ \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix} \end{matrix} \begin{matrix} \downarrow \text{column} \\ \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} \end{matrix}$

$$= \begin{bmatrix} (2 \times 3) + (1 \times 5) & (2 \times 2) + (1 \times 4) \\ (-3 \times 3) + (0 \times 5) & (-3 \times 2) + (0 \times 4) \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 8 \\ -9 & -6 \end{bmatrix}$$

Again $BA = \begin{matrix} \xrightarrow{\text{row}} \\ \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} \end{matrix} \begin{matrix} \downarrow \text{column} \\ \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix} \end{matrix}$

$$= \begin{bmatrix} (3 \times 2) + (2 \times -3) & (3 \times 1) + (2 \times 0) \\ (5 \times 2) + (4 \times -3) & (5 \times 1) + (4 \times 0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix}$$

Q.2 find AB

where $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 4 \\ 1 & 1 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 & 2 \\ -3 & 1 & 1 \\ -1 & 0 & 4 \end{bmatrix}$

Solution :- $AB = \begin{matrix} \xrightarrow{\text{row}} \\ \begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 4 \\ 1 & 1 & 5 \end{bmatrix} \end{matrix} \begin{matrix} \downarrow \text{column} \\ \begin{bmatrix} 2 & -1 & 2 \\ -3 & 1 & 1 \\ -1 & 0 & 4 \end{bmatrix} \end{matrix}$

$$= \begin{bmatrix} (2 \times 2) + (0 \times -3) + (1 \times -1) & (2 \times -1) + (0 \times 1) + (1 \times 0) & (2 \times 2) + (0 \times 1) + (1 \times 4) \\ (3 \times 2) + (-2 \times -3) + (4 \times -1) & (3 \times -1) + (-2 \times 1) + (4 \times 0) & (3 \times 2) + (-2 \times 1) + (4 \times 4) \\ (1 \times 2) + (1 \times -3) + (5 \times -1) & (1 \times -1) + (1 \times 1) + (5 \times 0) & (1 \times 2) + (1 \times 1) + (5 \times 4) \end{bmatrix}$$

$$= \begin{bmatrix} 4 + 0 + (-1) & (-2) + 0 + 0 & 4 + 0 + 4 \\ 6 + 6 + (-4) & (-3) + (-2) + 0 & 6 + (-2) + 16 \\ 2 + (-3) + (-5) & (-1) + 1 + 0 & 2 + 1 + 20 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & 8 \\ 8 & -5 & 20 \\ -6 & 0 & 23 \end{bmatrix}$$

Call III :- Multiplication of two non-square Matrices.

Q:-1

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 2 \end{bmatrix}_{2 \times 3} \quad B = \begin{bmatrix} 4 & 2 \\ -1 & 6 \\ 0 & 3 \end{bmatrix}_{3 \times 2}$$

Then evaluate AB and BA.

Solution :- Given $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 2 \\ -1 & 6 \\ 0 & 3 \end{bmatrix}$

Then $AB = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -1 & 6 \\ 0 & 3 \end{bmatrix}$

(Row of A) \times (Column of B)

$$= \begin{bmatrix} (2 \times 4) + (3 \times -1) + (-1 \times 0) & (2 \times 2) + (3 \times 6) + (-1 \times 0) \\ (4 \times 4) + (0 \times -1) + (2 \times 0) & (4 \times 2) + (0 \times 6) + (2 \times 3) \end{bmatrix}$$

$$= \begin{bmatrix} 8 + (-3) + 0 & 4 + 18 + (-3) \\ 16 + 0 + 0 & 8 + 0 + 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 19 \\ 16 & 14 \end{bmatrix}_{2 \times 2}$$

Again $BA = \begin{bmatrix} 4 & 2 \\ -1 & 6 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} (4 \times 2) + (2 \times 4) & (4 \times 3) + (2 \times 0) & (4 \times -1) + (2 \times 2) \\ (-1 \times 2) + (6 \times 4) & (-1 \times 3) + (6 \times 0) & (-1 \times -1) + (6 \times 2) \\ (0 \times 2) + (3 \times 4) & (0 \times 3) + (3 \times 0) & (0 \times -1) + (3 \times 2) \end{bmatrix}$$

$$= \begin{bmatrix} 8 + 8 & 12 + 0 & -4 + 4 \\ -2 + 24 & -3 + 0 & 1 + 12 \\ 0 + 12 & 0 + 0 & 0 + 6 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 12 & 0 \\ 22 & -3 & 13 \\ 12 & 0 & 6 \end{bmatrix}_{3 \times 3}$$

Imp Note :- To multiply an $m \times n$ matrix by $p \times q$ matrix, n must be equal to p (i.e. $n = p$) and resultant matrix will be of order $m \times q$.

Q:-2 If $A = \begin{bmatrix} 2 & 5 & 7 \\ -1 & 6 & 3 \end{bmatrix}_{2 \times 3}$ & $B = \begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix}_{3 \times 1}$

Then find AB and BA.

Solution :- $A = \begin{bmatrix} 2 & 5 & 7 \\ -1 & 6 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 5 & 7 \\ -1 & 6 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix}$$

(i) Here AB is possible
as no. of columns of A =
no. of rows of B = 3

$$= \begin{bmatrix} (2 \times 4) + (5 \times 9) + (7 \times -3) \\ (-1 \times 4) + (6 \times 9) + (3 \times -3) \end{bmatrix}$$

(ii) By multiplying
a 2×3 by a 3×1
we will get
a 2×1 matrix

$$= \begin{bmatrix} 8 + 45 + (-21) \\ (-4) + 54 + (-9) \end{bmatrix}$$

$$= \begin{bmatrix} 32 \\ 41 \end{bmatrix}_{2 \times 1}$$

$$\text{Again } BA = \begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} \begin{bmatrix} 2 & 5 & 7 \\ -1 & 6 & 3 \end{bmatrix}$$

Here BA is not possible.

as we have B is of order 3×1 &
A is of order 2×3

And number of columns of B is not equal
to the number of rows of A.

(iv) Division :-

Case I :- Division of a Matrix by a number.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a Matrix.

and k be a number.

$$\text{Then } \frac{A}{k} = \frac{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}{k} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \frac{1}{k}$$

NOTE

Instead of dividing k we can multiply $\frac{1}{k}$ (i.e. multiplicative inverse of k) to the Matrix.

$$= \begin{bmatrix} a(\frac{1}{k}) & b(\frac{1}{k}) \\ c(\frac{1}{k}) & d(\frac{1}{k}) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{k} & \frac{b}{k} \\ \frac{c}{k} & \frac{d}{k} \end{bmatrix}$$

Case II :- Division of a Matrix by another Matrix.

→ Technically there is no such thing as Matrix division.

→ Dividing a Matrix by another Matrix is Undefined.

So if $[A] \div [B]$ is undefined but we can solve it by multiplying inverse of Matrix $[B]$ with $[A]$

i.e. $[A] \times [B]^{-1}$, which is the closest equivalent of division.

How to find Multiplicative inverse of a Matrix :-

Let A be a Matrix, then its Multiplicative inverse is denoted by A^{-1}

and is obtained as $A^{-1} = \frac{\text{Adj}(A)}{|A|}$

Determinant :-

Basically Determinant is a scalar value ^{calculated} ~~derived~~ from the elements of a square Matrix.

Case I Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a order 2 square Matrix,

$$\text{Then } |A| = ad - bc$$

i.e. Multiplying the elements of main diagonal minus multiply elements of anti diagonal.

Case II

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be a order 3 square Matrix.

Then determinant of A is denoted as $|A|$.
and is defined as.

$$|A| = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - hf) - b(di - gf) + c(dh - eg).$$

- * To find the determinant of 3x3 Matrix first choose any row or any column.
- * Take the first element, cross out the row/column it belong to, find the determinant of the remaining 2x2 Matrix and multiply it with the chosen element.
- * While doing this calculation always refer to the Matrix sign chart.

i.e. $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

- * Repeat the process for the other two elements of the chosen row/column.

Q:-1 Find the determinant of the Matrix.

$$A = \begin{bmatrix} 4 & 2 \\ -7 & 8 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 4 & 2 \\ -7 & 8 \end{vmatrix}$$

$$= 32 - (-7 \times 2)$$

$$= 32 - (-14) = 32 + 14 = 46$$

Q:-2 Find the determinant of the Matrix

$$\begin{bmatrix} 1 & 4 & -2 \\ 3 & 4 & 9 \\ 5 & -2 & 8 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 4 & -2 \\ 3 & 4 & 9 \\ 5 & -2 & 8 \end{vmatrix}$$

Sign chart

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Choose R_1 to find the value of the above determinant.

$$= 1 \begin{vmatrix} 4 & 9 \\ -2 & 8 \end{vmatrix} - 4 \begin{vmatrix} 3 & 9 \\ 5 & 8 \end{vmatrix} + (-2) \begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix}$$

$$= 1(32 - (-18)) - 4(24 - 45) - 2(-6 - 20)$$

$$= 1(32 + 18) - 4(-21) - 2(-26) = 50 + 84 + 52 = 186 \text{ (Ans)}$$

Q.2 Find the determinant of the Matrix

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 5 & 0 & -6 \\ 9 & 3 & 7 \end{bmatrix}$$

Solution :-

$$|A| = \begin{vmatrix} 2 & -1 & 4 \\ 5 & 0 & -6 \\ 9 & 3 & 7 \end{vmatrix}$$

sign chart

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Choose C_1 to find the determinant of the

$$= 2 \begin{vmatrix} 0 & -6 \\ 3 & 7 \end{vmatrix} - 5 \begin{vmatrix} -1 & 4 \\ 3 & 7 \end{vmatrix} + 9 \begin{vmatrix} -1 & 4 \\ 0 & -6 \end{vmatrix}$$

$$= 2(0 - (-18)) - 5(-7 - 12) + 9(6 - 0)$$

$$= 2(18) - 5(-19) + 9(6)$$

$$= 36 + 95 + 54$$

$$= 185 \text{ (Ans.)}$$

Adjoint of a Matrix :-

Adjoint of a Matrix A is the Transpose of the co-factor Matrix of A .

$$\text{i.e. } \text{Adj}(A) = [\text{co-factor}(A)]^T$$

co-factor

co-factor of an element a_{ij} is denoted by C_{ij} (or A_{ij}) and is obtained as

$$C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

where M_{ij} is the Minor of the element a_{ij} .

Minor

"Minor of an element is obtained by eliminating the row and column it belongs to".

Case I - Consider a order 2 determinant.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2 \times 2}$$

$$\text{Minor of } a_{11}, M_{11} = a_{22}$$

$$\text{Minor of } a_{12}, M_{12} = a_{21}$$

$$\text{Minor of } a_{21}, M_{21} = a_{12}$$

$$\text{Minor of } a_{22}, M_{22} = a_{11}$$

Case II:- Consider a order 3 determinant.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Minor of } a_{11}, M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22} a_{33} - a_{32} a_{23}$$

$$\text{Minor of } a_{12}, M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21} a_{33} - a_{23} a_{31}$$

$$\text{Minor of } a_{13}, M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21} a_{32} - a_{22} a_{31}$$

$$\text{Minor of } a_{21}, M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12} a_{33} - a_{13} a_{32}$$

$$\text{Minor of } a_{22}, M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11} a_{33} - a_{13} a_{31}$$

$$\text{Minor of } a_{23}, M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11} a_{32} - a_{12} a_{31}$$

$$\text{Minor of } a_{31}, M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = a_{12} a_{23} - a_{13} a_{22}$$

$$\text{Minor of } a_{32}, M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = a_{11} a_{23} - a_{13} a_{21}$$

$$\text{Minor of } a_{33}, M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21}$$

Q.1 find the ^{Minor of} Matrix of $A = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}$

Solⁿ Given $A = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}$

$$M_{11} = 6$$

$$M_{12} = -1$$

$$M_{21} = 4$$

$$M_{22} = 2$$

Then Matrix of Minors

$$= \begin{bmatrix} 6 & -1 \\ 4 & 2 \end{bmatrix}$$

Q.2 Find the Minors of the Matrix

$$A = \begin{bmatrix} 5 & 4 & -2 \\ 2 & 0 & 9 \\ -3 & 6 & 7 \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} 0 & 9 \\ 6 & 7 \end{vmatrix} = 0 - 54 = -54$$

$$M_{12} = \begin{vmatrix} 2 & 9 \\ -3 & 7 \end{vmatrix} = 14 - (-27) = 14 + 27 = 41$$

$$M_{13} = \begin{vmatrix} 2 & 0 \\ -3 & 6 \end{vmatrix} = 12 - 0 = 12$$

$$M_{21} = \begin{vmatrix} 4 & -2 \\ 6 & 7 \end{vmatrix} = 28 - (-12) = 28 + 12 = 40$$

$$M_{22} = \begin{vmatrix} 5 & -2 \\ -3 & 7 \end{vmatrix} = 35 - 6 = 29$$

$$M_{23} = \begin{vmatrix} 5 & 4 \\ -3 & 6 \end{vmatrix} = 30 - (-12) = 30 + 12 = 42$$

$$M_{31} = \begin{vmatrix} 4 & -2 \\ 0 & 9 \end{vmatrix} = 36 - 0 = 36$$

$$M_{32} = \begin{vmatrix} 5 & -2 \\ 2 & 9 \end{vmatrix} = 45 - (-4) = 45 + 4 = 49$$

$$M_{33} = \begin{vmatrix} 5 & 4 \\ 2 & 0 \end{vmatrix} = 0 - 8 = -8$$

Then Matrix of Minors is given by

$$= \begin{bmatrix} -54 & 41 & 12 \\ 40 & 29 & 42 \\ 36 & 49 & -8 \end{bmatrix}$$

Co-factor = $(-1)^{i+j} M_{ij}$

So if we have to find co-factors for the same question, then

$$C_{11} = (-1)^{1+1} \cdot M_{11} = (-1)^2 \cdot (-54) = 1(-54) = -54$$

$$C_{12} = (-1)^{1+2} \cdot M_{12} = (-1)^3 \cdot 41 = (-1)(41) = -41$$

$$C_{13} = (-1)^{1+3} \cdot M_{13} = (-1)^4 \cdot 12 = 1(12) = 12$$

$$C_{21} = (-1)^{2+1} \cdot M_{21} = (-1)^3 \cdot 40 = (-1)40 = -40$$

$$C_{22} = (-1)^{2+2} \cdot M_{22} = (-1)^4 \cdot 29 = (1)(29) = 29$$

$$C_{23} = (-1)^{2+3} \cdot M_{23} = (-1)^5 \cdot 42 = (-1)(42) = -42$$

$$C_{31} = (-1)^{3+1} \cdot M_{31} = (-1)^4 \cdot 36 = 1 \cdot (36) = 36$$

$$C_{32} = (-1)^{3+2} \cdot M_{32} = (-1)^5 \cdot 49 = (-1)(49) = -49$$

$$C_{33} = (-1)^{3+3} \cdot M_{33} = (-1)^6 \cdot (-8) = 1(-8) = -8$$

Then the Matrix of co-factors is obtained as

$$\text{Co-factor}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -54 & -41 & 12 \\ -40 & 29 & -42 \\ 36 & -49 & -8 \end{bmatrix}$$

Shortcut :- (To find co-factor of order 3)

$$\begin{bmatrix} -54 & 41 & 12 \\ 40 & 29 & 42 \\ 36 & 49 & -8 \end{bmatrix} \sim \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \sim \begin{bmatrix} -54 & -41 & 12 \\ -40 & 29 & -42 \\ 36 & -49 & -8 \end{bmatrix}$$

(Matrix of Minors) (Matrix of co-factors)

(Just apply the checkerboard of minus to the Matrix of Minors)

$$\text{Adjoint} = [\text{co-factor}(A)]^T$$

So if we are going to find adjoint for the same question, then

As we already have the co-factor Matrix.

i.e.

$$\text{co-factor}(A) = \begin{bmatrix} -54 & -41 & 12 \\ -40 & 29 & -42 \\ 36 & -49 & -8 \end{bmatrix}$$

$$\text{Then } \text{Adj}(A) = [\text{co-factor}(A)]^T$$

$$= \begin{bmatrix} -54 & -40 & 36 \\ -41 & 29 & -49 \\ 12 & -42 & -8 \end{bmatrix}$$

Short-cut to find adjoint of a order two

Matrix:-

$$\text{let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{Then } \text{Adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

i.e. Just swap the positions of a and d
put negative in front of b and c.

Conclusion:-

So by using adjoint of a Matrix and its determinant we can find inverse of the Matrix using formula

$$\text{i.e. } \boxed{A^{-1} = \frac{\text{Adj}(A)}{|A|}}, |A| \neq 0.$$

If $|A| = 0$, such a Matrix is called Singular Matrix.

→ Inverse of a Matrix A is A^{-1}

$$\text{if } A A^{-1} = A^{-1} A = I$$

where I is the Identity Matrix.

→ Inverse is needed as Matrix division is not possible.

e.g. we want to find matrix X and matrix A and B are known.

$$\text{Given } XA = B.$$

But $X = B/A$ is not possible as we can't divide Matrices.

So we multiply both sides by A^{-1}

$$\Rightarrow X A A^{-1} = B A^{-1}$$

$$\Rightarrow X I = B A^{-1} \quad (\because A A^{-1} = I)$$

$$\Rightarrow X = B A^{-1}$$

By calculating A^{-1} & multiplying it with B we can easily find Matrix X .

Working Rule ~~to~~ to calculate the Inverse.

- step 1 - calculate the determinant.
- step 2 - calculate the Matrix of minors.
- step 3 - Turn that into Matrix of co-factors.
- step 4 - find Adjoint Matrix from co-factor Matrix.
- step 5 - Multiply that by $\frac{1}{\text{Determinant}}$.

Solution of system of linear Equations:-
(by Matrix Method)

System of linear equation means when we have two or more linear equations working together.

Suppose we have the following system of eq's.

$$a_1 x + b_1 y + c_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3$$

Then the above system of eq's can be written in Matrix form i.e. $AX = B$

$$\text{Where } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Known as coefficient Matrix.

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{Matrix of unknowns}$$

$$\text{and } B = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

As we have to find solⁿ i.e. values of x, y, z , we need to find Matrix 'X'

$$\text{as } AX = B$$

$$\Rightarrow X = BA^{-1}$$

$$\text{or } X = A^{-1}B$$

$$= \frac{\text{Adj}(A)}{|A|} \cdot B$$

Q.1 solve by Matrix Method.

$$x + 2y = 3, \quad 3x + y = 4$$

Solution:- Given linear eq^s are

$$x + 2y = 3$$

$$3x + y = 4$$

The above system of eq^s can be written in Matrix form i.e.

$$AX = B$$

$$\text{where } A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\text{Then } X = A^{-1}B \quad \text{--- (1)}$$

Now find A^{-1}

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 = -5 \neq 0$$

$\Rightarrow A^{-1}$ exists.

$$C_{11} = (-1)^{1+1} \cdot M_{11} = (1)^2 \cdot 1 = 1$$

$$C_{12} = (-1)^{1+2} \cdot M_{12} = (-1)^3 \cdot 3 = -3$$

$$C_{21} = (-1)^{2+1} \cdot M_{21} = (-1)^3 \cdot 2 = -2$$

$$C_{22} = (-1)^{2+2} \cdot M_{22} = (-1)^4 \cdot 1 = 1$$

$$\text{Then co-factor } (A) = \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$$

$$\text{Adj}(A) = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}}{-5}$$

Putting these values in eqⁿ ①

$$\Rightarrow X = A^{-1} B = \frac{\text{Adj}(A)}{|A|} \cdot B$$

$$\Rightarrow X = \frac{\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix}}{-5}$$

$$\Rightarrow X = \frac{\begin{bmatrix} 3+(-8) \\ -9+4 \end{bmatrix}}{-5}$$

$$\Rightarrow X = \begin{bmatrix} -5 \\ -5 \end{bmatrix} \times \frac{1}{-5}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

i.e. $x=1$ & $y=1$

Q.2 Solve $x-y+z=4$
 $2x+y-3z=0$
 $x+y+z=2$

Solution:- The above system of equations can be written in $AX=B$ form

$$\text{where } A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\text{Then } X = A^{-1} B \quad \text{--- ①}$$

To find A^{-1}

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 1(1+3) + 1(2+3) + 1(2-1) = 4+5+1 = 10 \neq 0$$

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^0 \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = 1(1 - (-3)) = 1+3 = 4$$

$$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = (-1)(2 - (-3)) = -1(5) = -5$$

$$C_{13} = (-1)^{1+3} M_{13} = (-1)^4 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = (+1)(2-1) = 1$$

$$C_{21} = (-1)^{2+1} M_{21} = (-1)^3 \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = (-1)(-1-1) = (-1)(-2) = 2$$

$$C_{22} = (-1)^{2+2} M_{22} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1(1-1) = 0$$

$$C_{23} = (-1)^{2+3} M_{23} = (-1)^5 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = (-1)(1 - (-1)) = (-1)(2) = -2$$

$$C_{31} = (-1)^{3+1} M_{31} = (-1)^4 \begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = 1(3-1) = 2$$

$$C_{32} = (-1)^{3+2} M_{32} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = (-1)(-3-2) = (-1)(-5) = 5$$

$$C_{33} = (-1)^{3+3} M_{33} = (-1)^6 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = (1)(1 - (-2)) = 1+2 = 3$$

$$\text{co-factor}(A) = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$$

$$\text{Adj}(A) = [\text{co-factor}(A)]^T$$

$$= \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

putting these values in eqⁿ (1)

$$X = A^{-1} B$$

$$= \frac{\text{Adj}(A)}{|A|} \cdot B$$

$$= \frac{\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}}{10} \cdot \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4+0+6 \end{bmatrix} \times \frac{1}{10}$$

$$= \begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix} \times \frac{1}{10}$$

$$= \begin{bmatrix} 20/10 \\ -10/10 \\ 10/10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

So we have

$$X = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow x=2, y=-1 \text{ and } z=1$$

which is the required solⁿ.

Important 5 Marks Questions:-

Q.1 find the value of x and y

$$\text{when } \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Solution:- Given $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x+3y \\ 2x-y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\text{i.e. } x+3y=4 \text{ --- (I)}$$

$$2x-y=1 \text{ --- (II)}$$

Solving (I) & (II)

$$\text{eq (I)} \times 2 \Rightarrow 2x+6y=8$$

$$\text{eq (II)} \times 1 \Rightarrow \begin{array}{r} 2x-y=1 \\ (-) \quad (+) \quad (+) \\ \hline \end{array}$$

$$7y=7$$

$$\Rightarrow y = 7/7 = 1$$

$\boxed{y=1}$, putting value of $y=1$ in

eq (I) we have $\boxed{x=1}$

Q.2 Verify $(AB)^T = B^T A^T$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix}$

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$$

Solution :- L.H.S $AB = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+4+(-3) & 2+0+3 \\ 3+(-2)+(-1) & 6+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 \\ -2 & 7 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 2 & -2 \\ 5 & 7 \end{bmatrix}$$

R.H.S $B^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \end{bmatrix}$

$$B^T A^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+(-3) & 3+(-4)+(-1) \\ 2+0+3 & 6+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ 5 & 7 \end{bmatrix}$$

L.H.S = R.H.S (proved)

Q.3 If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ and I is the 2×2 unit matrix. Find $(A-2I)(A-3I)$.

Solution Given $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then $A-2I = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4-2 & 2-0 \\ -1-0 & 1-2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$

Again $A-3I$

$$= \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & 2-0 \\ -1-0 & 1-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

Now $(A-2I)(A-3I)$

$$= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+(-2) & 4+(-4) \\ -1+(1) & -2+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Q.4 solve by Matrix Method.

$$2x - y + z = 0$$

$$3x + 4y - z = 0$$

Solution:- Given linear equations are

$$2x - y = -2$$

$$3x + 4y = 3$$

The above system of linear eq's can be written in Matrix form -

$$\text{i.e. } AX = B$$

$$\text{where } A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\text{Then } X = A^{-1}B$$

$$= \frac{\text{Adj}(A)}{|A|} \cdot B$$

$$= \frac{\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}}{8 - (-3)} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix} \cdot \frac{1}{11}$$

$$= \begin{bmatrix} -5 \\ 12 \end{bmatrix} \times \frac{1}{11}$$

$$= \begin{bmatrix} -5/11 \\ 12/11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5/11 \\ 12/11 \end{bmatrix}$$

So $x = -5/11$ & $y = 12/11$ is the required solution.

Q.5 If $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

Verify $A^2 - 3A + 2I = 0$

Solution $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

Then $A^2 = A \cdot A$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & -2+0+(-4) \\ 2+4+0 & 0+4+0 & -4+8+8 \\ 0+0+0 & 0+0+0 & 0+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix}$$

$$3A = 3 \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & -6 \\ 6 & 6 & 12 \\ 0 & 0 & 6 \end{bmatrix}$$

$$2I = 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Consider L.H.S

$$A^2 - 3A + 2I$$

$$= \begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 & -6 \\ 6 & 6 & 12 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3+2 & 0-0+0 & -6+6+0 \\ 6-6+0 & 4-6+2 & 12-12+0 \\ 0-0+0 & 0-0+0 & 4-6+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{R.H.S}$$

(proved).

Properties of determinant :-

① Property of Reflection

The determinant remains unaltered if its rows are changed into columns and the columns into rows.

$$\text{eg:- } \Delta = \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} = 12 - 10 = 2$$

$$\text{using Property, } \Delta' = \begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix} = 12 - 10 = 2$$

② All-Zero property

If all the elements of a row or column are zero then the determinant is zero.

$$\text{eg:- } \Delta = \begin{vmatrix} 4 & 5 & 0 \\ 2 & -3 & 0 \\ 9 & 4 & 0 \end{vmatrix} = 0$$

③ Proportionality (Repetition) Property

If all the elements of a row (column) are proportional (identical) to the elements of

Same other row (or column), then the determinant is zero. e.g.

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & -8 \\ 1 & 2 & 3 \end{vmatrix} = 0 \quad \left(\begin{array}{l} \text{as } R_1 \text{ is} \\ \text{identical to} \\ R_3 \end{array} \right)$$

④ Switching Property

The interchange of any two rows (or columns), if a determinant changes its sign.

$$\Delta = \begin{vmatrix} 2 & -1 & 4 \\ 2 & 0 & 3 \\ 4 & 1 & 0 \end{vmatrix}$$

$$= 4 \begin{vmatrix} -1 & 4 \\ 0 & 3 \end{vmatrix} - 1 \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix}$$

$$= 4(-3-0) - 1(6-8)$$

$$= -12 - (-2) = -12 + 2 = -10$$

Using Property
By $\begin{matrix} R_1 \leftrightarrow R_2 \end{matrix}$

$$\Delta' = \begin{vmatrix} 2 & 0 & 3 \\ 2 & -1 & 4 \\ 4 & 1 & 0 \end{vmatrix}$$

$$= 2 \begin{vmatrix} -1 & 4 \\ 1 & 0 \end{vmatrix} + 3 \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix}$$

$$= 2(0-4) + 3(2+4)$$

$$= -8 + 18 = 10$$

⑤ Scalar Multiple Property

If all the elements of a row (or column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.

$$\text{e.g. } \Delta = \begin{vmatrix} 4 & 2 \\ -1 & 3 \end{vmatrix} \\ = 12 - (-2) = 14$$

By using Property $C_1 \rightarrow 2C_1$

$$\Delta' = \begin{vmatrix} 8 & 2 \\ -2 & 3 \end{vmatrix}$$

$$= 24 - (-4)$$

$$= 28$$

⑥ Sum property:

If each element in any row or column consists of two or more terms, then the determinant can be expressed as the sum of two or more than two determinants.

$$\text{e.g. } \Delta = \begin{vmatrix} a_1+b_1 & c_1 & d_1 \\ a_2+b_2 & c_2 & d_2 \\ a_3+b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

① Property of Invariance :-

If the elements of any row (or column) be increased or decreased by the same scalar with the corresponding elements of another row (or column) then the determinant remains unaltered.

e.g.

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 4 & 0 & 3 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 1(0-3) - 2(8-3) - 1(4-0)$$

$$= -3 - 10 - 4 = -17$$

using property

$$R_1 \rightarrow R_1 + 2R_3$$

$$\Delta' = \begin{vmatrix} 3 & 4 & 3 \\ 4 & 0 & 3 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} - 4 \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} + 3 \begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 3(0-3) - 4(8-3) + 3(4-0) = -9 - 20 + 12 = -27 + 12 = -15$$

Q:-1 Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Proof :- L.H.S

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$C_1 \rightarrow C_1 - C_a \quad \& \quad C_2 \rightarrow C_2 - C_3$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a-b)(a+b) & (b-c)(b+c) & c^2 \end{vmatrix}$$

$$= (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c) \left\{ 1 \left| \begin{array}{cc} 1 & 1 \\ a+b & b+c \end{array} \right| \right\}$$

$$= (a-b)(b-c) \{ (b+c) - (a+b) \}$$

$$= (a-b)(b-c) \{ b+c-a-b \}$$

$$= (a-b)(b-c)(c-a) \quad \text{(proved)}$$

② Prove that $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz \left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)$

Proof:- L.H.S

$$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}$$

$$C_1 \rightarrow C_1 - C_2, \quad C_2 \rightarrow C_2 - C_3$$

$$= \begin{vmatrix} x & 0 & 1 \\ -y & y & 1 \\ 0 & -z & 1+z \end{vmatrix}$$

$$= \begin{vmatrix} x & 0 & 1 \\ -y & y & 1 \\ 0 & -z & 1+z \end{vmatrix}$$

Expand through R_1

$$= x \begin{vmatrix} y & 1 \\ -z & 1+z \end{vmatrix} + 1 \begin{vmatrix} -y & y \\ 0 & -z \end{vmatrix}$$

$$= x \{ (y)(1+z) - (-z)(1) \} + 1 \{ (-y)(-z) - 0 \}$$

$$= x \{ y + yz + z \} + yz$$

$$= xy + xyz + xz + yz$$

$$= xyz \left(\frac{1}{x} + 1 + \frac{1}{y} + \frac{1}{z} \right)$$

$$= xyz \left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) = \text{R.H.S} \quad \text{(proved)}$$

Method

Step-1 Try to make factors, then take it common.

Step-2 Try to make zeros (as much as you can)

Step-3 Expand.

Q.3 Prove that

$$\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$$

Proof

L.H.S

$$\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix}$$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} x+a+b+c & b & c \\ a+x+b+c & x+b & c \\ a+b+x+c & b & x+c \end{vmatrix}$$

Taking $(x+a+b+c)$ common from C_1

$$= (x+a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{vmatrix}$$

$$R_1 \rightarrow R_1 - R_2 \quad \& \quad R_2 \rightarrow R_2 - R_3$$

$$= (x+a+b+c) \begin{vmatrix} 0 & -x & 0 \\ 0 & x & -x \\ 1 & b & x+c \end{vmatrix}$$

$$R_2 \rightarrow R_1 + R_2$$

$$= (x+a+b+c) \begin{vmatrix} 0 & -x & 0 \\ 0 & 0 & -x \\ 1 & b & x+c \end{vmatrix}$$

Expand by taking C_1

$$= (x+a+b+c) \left\{ 1 \begin{vmatrix} -x & 0 \\ 0 & -x \end{vmatrix} \right\}$$

$$= (x+a+b+c) (x^2 - 0)$$

$$= x^2(x+a+b+c) = \text{R.H.S}$$

(proved)

Q.4 Prove that
$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

Proof :-

L.H.S

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$$

$$C_1 \rightarrow C_1 - C_2 \quad \text{and} \quad C_2 \rightarrow C_2 - C_3$$

$$= \begin{vmatrix} a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \\ bc-ca & ca-ab & ab \end{vmatrix}$$

$$= \begin{vmatrix} a-b & b-c & c \\ (a+b)(a-b) & (b+c)(b-c) & c^2 \\ -c(a-b) & -a(b-c) & ab \end{vmatrix}$$

=
$$\begin{vmatrix} 1 & 1 & c \\ a+b & b+c & c^2 \\ -c & -a & ab \end{vmatrix}$$

Taking $(a-b)$ common from C_1
and $(b-c)$ common from C_2

$$= (a-b)(b-c) \begin{vmatrix} 1 & 1 & c \\ a+b & b+c & c^2 \\ -c & -a & ab \end{vmatrix}$$

$$C_1 \rightarrow C_1 - C_2$$

$$= (a-b)(b-c) \begin{vmatrix} 0 & 1 & c \\ a-c & b+c & c^2 \\ a-c & -a & ab \end{vmatrix}$$

$$= (a-b)(b-c) \begin{vmatrix} 0 & 1 & c \\ -(c-a) & b+c & c^2 \\ -(c-a) & -a & ab \end{vmatrix}$$

Taking $(c-a)$ common from C_1

$$= (a-b)(b-c)(c-a) \begin{vmatrix} 0 & 1 & c \\ -1 & b+c & c^2 \\ -1 & -a & ab \end{vmatrix}$$

~~$R_2 \rightarrow R_2 + R_3$~~ $R_2 \rightarrow R_2 - R_3$

$$= (a-b)(b-c)(c-a) \begin{vmatrix} 0 & 1 & c \\ 0 & b+c+a & c^2-ab \\ -1 & -a & ab \end{vmatrix}$$

Expand by taking (choosing) C_1

$$= (a-b)(b-c)(c-a) \left\{ -1 \begin{vmatrix} 1 & c \\ b+c+a & c^2-ab \end{vmatrix} \right\}$$

$$= (a-b)(b-c)(c-a) \left\{ -1 (c^2-ab - bc + c^2-ac) \right\}$$

$$= (a-b)(b-c)(c-a) (ab+bc+ca) = \text{R.H.S. (proved)}$$

Q:5 Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Proof: - L.H.S

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$R_1 \rightarrow R_1 + R_2 + R_3$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$C_1 \rightarrow C_1 - C_2 \quad \& \quad C_2 \rightarrow C_2 - C_3$$

$$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ a+b+c & -(a+b+c) & 2b \\ 0 & a-b+c & c-a-b \end{vmatrix}$$

Expand by choosing R_1

$$= (a+b+c) \left\{ 1 \begin{vmatrix} a+b+c & -(a+b+c) \\ 0 & a+b+c \end{vmatrix} \right\}$$

$$= (a+b+c) \left\{ (a+b+c)^2 - 0 \right\}$$

$$= (a+b+c)^3 = \text{R.H.S (proved)}$$

Q.6 :- If $A+B+C = \pi$, P.T. $\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix} = 0$

L.H.S :-

$$\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$$

$$R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$$

$$= \begin{vmatrix} \sin^2 A - \sin^2 B & \cot A - \cot B & 0 \\ \sin^2 B - \sin^2 C & \cot B - \cot C & 0 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$$

Expand by choosing C_3

$$= 1 \begin{vmatrix} \sin^2 A - \sin^2 B & \cot A - \cot B \\ \sin^2 B - \sin^2 C & \cot B - \cot C \end{vmatrix}$$

$$= (\sin^2 A - \sin^2 B)(\cot B - \cot C) - (\sin^2 B - \sin^2 C)(\cot A - \cot B)$$

$$= \left\{ \sin(A+B) \cdot \sin(A-B) \right\} \left\{ \frac{\cos B}{\sin B} - \frac{\cos C}{\sin C} \right\} - \left\{ \sin(B+C) \cdot \sin(B-C) \right\} \left\{ \frac{\cos A}{\sin A} - \frac{\cos B}{\sin B} \right\}$$

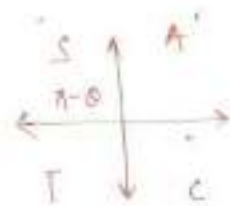
$$= \sin(A+B) \cdot \sin(A-B) \left\{ \frac{\cos B \cdot \sin C - \cos C \cdot \sin B}{\sin B \cdot \sin C} \right\} - \sin(B+C) \cdot \sin(B-C) \left\{ \frac{\cos A \cdot \sin B - \cos B \cdot \sin A}{\sin A \cdot \sin B} \right\}$$

$$= \frac{\sin(A+B) \sin(A-B) \cdot \sin(C-B)}{\sin B \cdot \sin C} - \frac{\sin(B+C) \cdot \sin(B-C) \cdot \sin(B-A)}{\sin A \cdot \sin B}$$

$$A+B+C=\pi$$

$$\rightarrow A+B=\pi-C$$

$$\rightarrow \sin(A+B) = \sin(\pi-C) \\ = \sin C$$



Similarly $\sin(B+C) = \sin A$

$$\frac{\cancel{\sin C} \cdot \sin(A-B) \cdot \sin(C-B)}{\sin B \cdot \cancel{\sin C}} = \frac{\cancel{\sin A} \cdot \sin(B-A) \cdot \sin(B-C)}{\sin B \cdot \cancel{\sin A}}$$

$$= \frac{\sin(A-B) \cdot \sin(C-B)}{\sin B} = \frac{\sin(B-A) \cdot \sin(B-C)}{\sin B}$$

$$= \frac{\sin\{- (B-A)\} \cdot \sin\{- (B-C)\}}{\sin B} = \frac{\sin(B-A) \cdot \sin(B-C)}{\sin B}$$

($\because \sin(-\theta) = -\sin(\theta)$)

$$= \frac{-\sin(B-A) \cdot -\sin(B-C)}{\sin B} = \frac{\sin(B-A) \cdot \sin(B-C)}{\sin B}$$

$$= \frac{\sin(B-A) \cdot \sin(B-C)}{\sin B} = \frac{\sin(B-A) \cdot \sin(B-C)}{\sin B}$$

$$= 0 = R.H.S. \text{ (proved)}$$

Solution of Linear System of Equations using Determinant:-

Let's consider the following System of linear equations.

$$\text{ie. } a_1x + b_1y = c_1 \quad \text{--- (1)}$$

$$a_2x + b_2y = c_2 \quad \text{--- (2)}$$

To solve 'x'

$$\text{eq (1)} \times b_2 \rightarrow b_2 a_1 x + b_2 b_1 y = b_2 c_1$$

$$\text{eq (2)} \times b_1 \rightarrow b_1 a_2 x + b_1 b_2 y = b_1 c_2$$

$$\Rightarrow b_2 a_1 x - b_1 a_2 x = b_2 c_1 - b_1 c_2$$

$$\Rightarrow (b_2 a_1 - b_1 a_2) x = b_2 c_1 - b_1 c_2$$

$$\Rightarrow x = \frac{b_2 c_1 - b_1 c_2}{b_2 a_1 - b_1 a_2}$$

To solve 'y'

$$\text{eq (1)} \times a_2 \rightarrow a_2 a_1 x + a_2 b_1 y = a_2 c_1$$

$$\text{eq (2)} \times a_1 \rightarrow a_1 a_2 x + a_1 b_2 y = a_1 c_2$$

$$\Rightarrow (a_2 b_1 - a_1 b_2) y = a_2 c_1 - a_1 c_2$$

$$\Rightarrow y = \frac{a_2 c_1 - a_1 c_2}{a_2 b_1 - a_1 b_2}$$

So the required solutions are

$$x = \frac{b_2 c_1 - b_1 c_2}{a_1 b_2 - a_2 b_1}, \quad y = \frac{a_2 c_1 - a_1 c_2}{a_2 b_1 - a_1 b_2}$$

We got the above solution by using elimination method, but the above can be represented by Δ in the form of determinant.

$$\Rightarrow x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad * \quad y = \frac{-(a_1 c_2 - a_2 c_1)}{-(a_1 b_2 - a_2 b_1)}$$

$$= \frac{a_2 c_2 - a_1 c_1}{a_1 b_2 - a_2 b_1}$$

$$= \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

~~above~~ So this is the solⁿ we are getting by using ~~above~~ determinant, is known as Cramer's Rule

So Cramer's Rule is a Method that uses determinants to solve system of linear eqⁿ's.

which introduces new notations also.

\rightarrow we can notice that the denominator ^{of} both x & y is the determinant of coefficient Matrix.

and Δ = Determinant of the coefficient Matrix.

Δ_x = Determinant of coefficient Matrix in which the x -column is replaced by the constant column.

* and this is the numerator in the solution of x .

Δ_y = Determinant of coefficient Matrix in which the y -column is replaced by the constant column.

* and this the numerator in the solution of y .

So finally we can say

$$\boxed{x = \frac{\Delta_x}{\Delta}}, \quad \boxed{y = \frac{\Delta_y}{\Delta}}$$

Q.1 Solve $2x - y = 2$
 $3x + y = 13$ using cramer's rule.

Solution Given $2x - y = 2$
 $3x + y = 13$

$$\text{Then } \Delta = \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2 - (-3) = 2 + 3 = 5$$

$$\Delta_x = \begin{vmatrix} 2 & -1 \\ 13 & 1 \end{vmatrix} = 2 - (-13) = 15$$

$$\Delta_y = \begin{vmatrix} 2 & 2 \\ 3 & 13 \end{vmatrix} = 26 - 6 = 20$$

$$\text{So } x = \frac{\Delta_x}{\Delta} = \frac{15}{5} = 3$$

$$y = \frac{\Delta_y}{\Delta} = \frac{20}{5} = 4$$

So the required solⁿ is $x=3, y=4$

Q.2 Solve $x + y + z = 3$
 $2x + 3y + 4z = 9$
 $x + 2y - 4z = -1$ using cramer's Rule.

Solution Given $x + y + z = 3$
 $2x + 3y + 4z = 9$
 $x + 2y - 4z = -1$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 3 & 4 \\ 2 & -4 \end{vmatrix} - 1 \begin{vmatrix} 2 & 4 \\ 1 & -4 \end{vmatrix} + 1 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= 1(-12-8) - 1(-8-4) + 1(4-3)$$

$$= -20 - 1(-12) + 1$$

$$= -20 + 12 + 1 = -7$$

$$\Delta_x = \begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 4 \\ -1 & 2 & -4 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 3 & 4 \\ 2 & -4 \end{vmatrix} - 1 \begin{vmatrix} 9 & 4 \\ -1 & -4 \end{vmatrix} + 1 \begin{vmatrix} 9 & 3 \\ -1 & 2 \end{vmatrix}$$

$$= 3(-12-8) - 1(-36+4) + 1(18-(-3))$$

$$= -60 + 32 + 21$$

$$= -7$$

$$\Delta_y = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 9 & 4 \\ 1 & -1 & -4 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 9 & 4 \\ -1 & -4 \end{vmatrix} - 3 \begin{vmatrix} 2 & 4 \\ 1 & -4 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1(-36+4) - 3(-8-4) + 1(-2-1)$$

$$= -32 + 36 - 11$$

$$= -7$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 3 & 9 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 3 & 9 \\ 2 & -1 \end{vmatrix} - 1 \begin{vmatrix} 2 & 9 \\ 1 & -1 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= (-3-18) - 1(-2-9) + 3(4-3)$$

$$= (-21) + 11 + 3$$

$$= -7$$

$$\text{So } x = \frac{\Delta_x}{\Delta} = \frac{-7}{-7} = 1$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-7}{-7} = 1$$

$$z = \frac{\Delta_z}{\Delta} = \frac{-7}{-7} = 1$$

So the required solⁿ is $x=1, y=1, z=1$.

TRIGONOMETRY

→ The word trigonometry comes from three greek words.

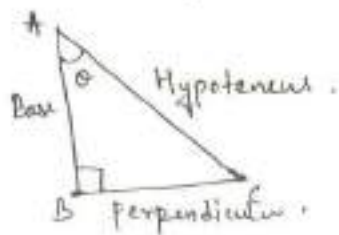
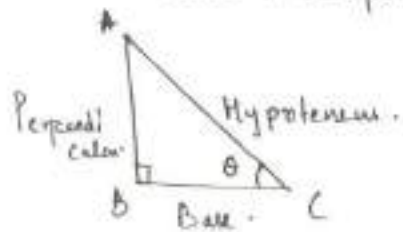
TRI → Three

GON → Sides

METRON - Measurement.

So Trigonometry means ~~most~~ measurement of three sides of basically triangle (i.e. right angled triangle).

NOTE :- ① There are three sides of triangle namely Base, Perpendicular and Hypotenuse.



i.e. Longest side = Hypotenuse (H)
Side in front of angle θ = perpendicular (P)
Remaining one side = Base (B)

② If B, P, H are three sides of right angled triangle.

$$\text{Then } \boxed{P^2 + B^2 = H^2} \text{ known as}$$

Pythagoras theorem.

③ By taking ratios of sides we have.

$$\frac{P}{H}, \frac{B}{H}, \frac{H}{P}, \frac{H}{B}, \frac{P}{B}, \frac{B}{P} \text{ (i.e. 6 ratios)}$$

Known as trigonometric ratios.

In mathematics the above ratios have some specific names as follows:

Trigonometric Ratios

| | |
|-----------------------------|-----------------------------|
| $\sin \theta = \frac{P}{H}$ | $\csc \theta = \frac{H}{P}$ |
| $\cos \theta = \frac{B}{H}$ | $\sec \theta = \frac{H}{B}$ |
| $\tan \theta = \frac{P}{B}$ | $\cot \theta = \frac{B}{P}$ |

from the above trigonometric Ratios.
we can derive the followings:

$$\textcircled{1} \sin \theta = \frac{1}{\operatorname{cosec} \theta}$$

$$\text{or } \operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

$$\textcircled{2} \cos \theta = \frac{1}{\sec \theta}$$

$$\text{or } \sec \theta = \frac{1}{\cos \theta}$$

$$\textcircled{3} \tan \theta = \frac{1}{\cot \theta}$$

$$\text{or } \cot \theta = \frac{1}{\tan \theta}$$

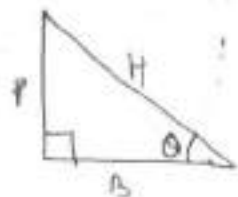
$$\textcircled{4} \tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

Pythagoras Theorem:-

$$P^2 + B^2 = H^2$$

$$\Rightarrow \frac{P^2}{H^2} + \frac{B^2}{H^2} = 1$$



$$\Rightarrow \left(\frac{P}{H}\right)^2 + \left(\frac{B}{H}\right)^2 = 1$$

$$\Rightarrow (\sin \theta)^2 + (\cos \theta)^2 = 1$$

$$\Rightarrow \boxed{\sin^2 \theta + \cos^2 \theta = 1}$$

$$\text{or } \sin^2 \theta = 1 - \cos^2 \theta$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

Dividing $\sin^2 \theta$ in formula (5)

$$\boxed{1 + \cot^2 \theta = \operatorname{cosec}^2 \theta}$$

Dividing $\cos^2 \theta$ in (5)

$$\boxed{\tan^2 \theta + 1 = \sec^2 \theta}$$

Angles:-

→ Angles basically denoted by $\theta, \alpha, \beta, \dots$

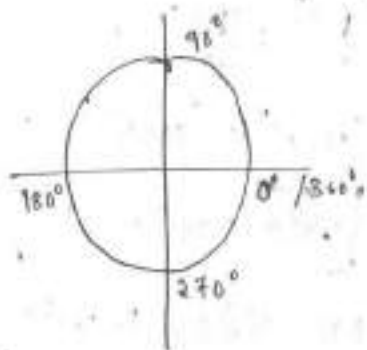
measured in anti-clockwise direction.

If measured in clockwise direction the angle will be negative angle.

→ There are two units to measure angle.

(i) Degree (ii) Radian

| Degree | Radian |
|-------------|----------|
| 0° | 0 |
| 30° | $\pi/6$ |
| 45° | $\pi/4$ |
| 60° | $\pi/3$ |
| 90° | $\pi/2$ |
| 180° | π |
| 270° | $3\pi/2$ |
| 360° | 2π |



$\theta = 30^\circ$
 $30^\circ - 360^\circ = -330^\circ$
 $-330^\circ - 360^\circ = -690^\circ$

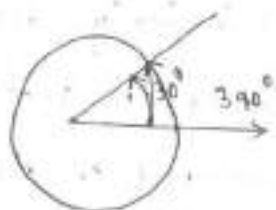
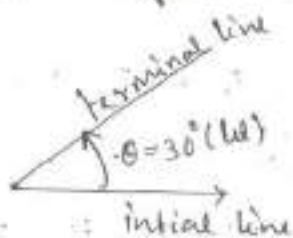
co-terminal angles.

Trigonometric Ratios of some standard angles.

| | 0° | 30° | 45° | 60° | 90° |
|-------|-----------|--------------|--------------|--------------|------------|
| sin | 0 | $1/2$ | $1/\sqrt{2}$ | $\sqrt{3}/2$ | 1 |
| cos | 1 | $\sqrt{3}/2$ | $1/\sqrt{2}$ | $1/2$ | 0 |
| tan | 0 | $1/\sqrt{3}$ | 1 | $\sqrt{3}$ | ∞ |
| cot | ∞ | $\sqrt{3}$ | 1 | $1/\sqrt{3}$ | 0 |
| sec | 1 | $2/\sqrt{3}$ | $\sqrt{2}$ | 2 | ∞ |
| cosec | ∞ | 2 | $\sqrt{2}$ | $2/\sqrt{3}$ | 1 |

Co-terminal angles.

If the terminal lines of two angles are same then, the angles are known as co-terminal angles.



$\theta = 30^\circ$
 $30^\circ + 360^\circ = 390^\circ$
 $390^\circ + 360^\circ = 750^\circ$

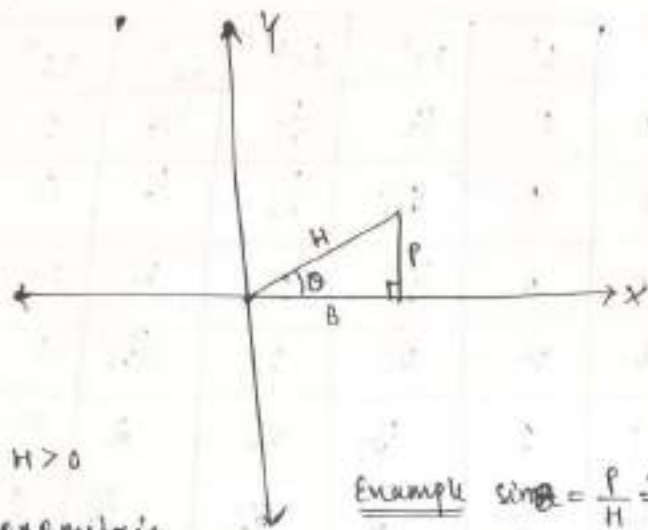
are co-terminal angles.

Trigonometric functions:-

There are 6 trigonometric functions.

$\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$ and $\csc x$

which are represented in two dimensional plane.



1st Quadrant

$$B > 0, P > 0 \text{ \& } H > 0$$

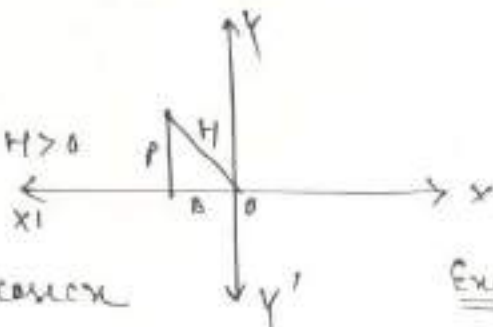
So all the trigonometric functions are +ve.

Example $\sin \theta = \frac{P}{H} = \frac{+}{+} = +ve$

$\sec \theta = \frac{H}{B} = \frac{+ve}{+ve} = +ve$

2nd Quadrant

$$B < 0, P > 0 \text{ \& } H > 0$$



So \sin and $\csc x$ are +ve and other functions are -ve.

Example

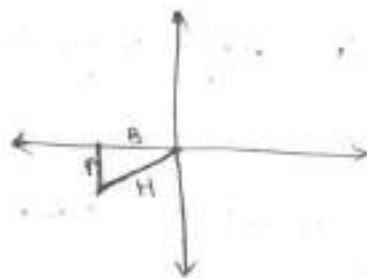
$$\sin \theta = \frac{P}{H} = \frac{+ve}{+ve} = +ve$$

$$\tan \theta = \frac{P}{B} = \frac{+ve}{-ve} = -ve$$

3rd Quadrant

In 3rd quadrant

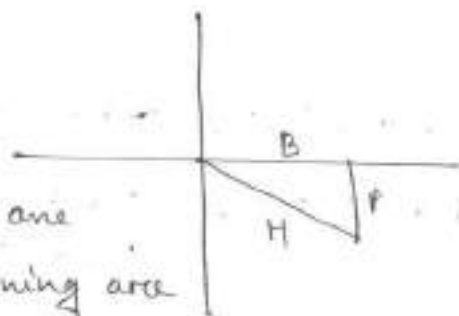
$$B < 0, P < 0 \text{ \& } H > 0$$



So \tan and \cot are +ve and remaining functions are -ve.

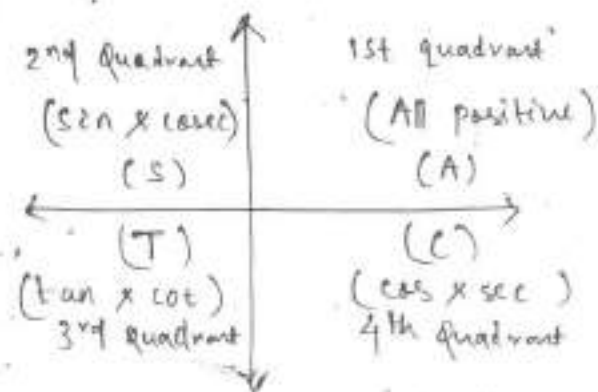
4th Quadrant

$$P < 0, B > 0, H > 0$$



So \cos & \sec are +ve and remaining are -ve.

So finally.



ASTC Rule.

Conversion of any angle into acute angle :-

Step 1 Convert the given angle into the following format.

$$\begin{array}{cccc} 90 + \theta & 180 - \theta & 270 - \theta & 360 - \theta \\ 90 - \theta & 180 + \theta & 270 + \theta & 360 + \theta \end{array}$$

Step 2 For $180^\circ/360^\circ \Rightarrow$ ~~ratios~~ functions same.

$$\text{For } 90^\circ/270^\circ \Rightarrow \sin \leftrightarrow \cos$$

$$\tan \leftrightarrow \cot$$

$$\sec \leftrightarrow \operatorname{cosec}$$

Step 3 Then put the sign depending upon the original function.

Example

$$\sin 210^\circ$$

$$= \sin (180^\circ + 30^\circ)$$

$$= -\sin 30^\circ$$

$$= -\frac{1}{2}$$

($\rightarrow 180^\circ + 30^\circ$ will lie on 3rd quadrant. So put (-ve) sign as sin function is -ve on 3rd quadrant)

Domain and Range of Trigonometric Functions

① $f(x) = \sin x$

Here x can be any angle $-\infty$ to ∞

So Domain = \mathbb{R}

$\forall x \in \mathbb{R}$, $\sin x = \frac{P}{H}$ where $H \geq P$ and

then Range will be $[-1, 1]$

② $f(x) = \cos x$

Here x can be any angle $-\infty$ to ∞

So Domain = \mathbb{R}

$\cos x = \frac{B}{H}$ where $H \geq B$

then Range will be $[-1, 1]$

③ $f(x) = \tan x = \frac{\sin x}{\cos x}$, $\cos x \neq 0$

i.e. $x \neq \pi/2, 3\pi/2, 5\pi/2, \dots$

i.e. $x \neq (2n+1)\pi/2$

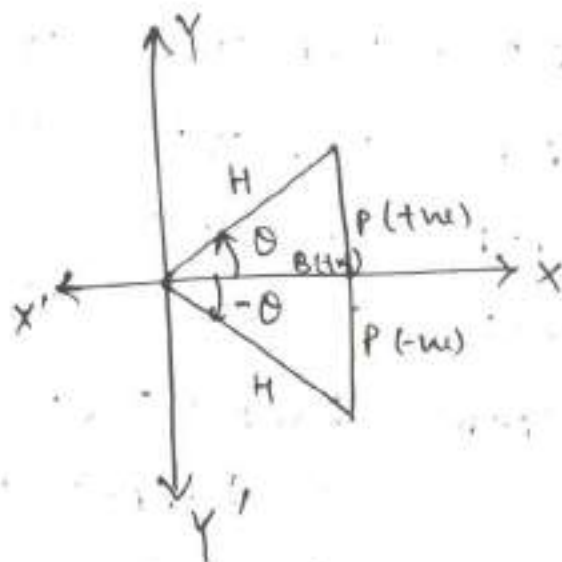
Domain = $\mathbb{R} - \{(2n+1)\pi/2, n \in \mathbb{Z}\}$

$\tan x = \frac{P}{B}$ so Range = \mathbb{R}

| Function | Domain | Range |
|--------------------------|--|----------------------------------|
| $\sin x$ | \mathbb{R} | $[-1, 1]$ |
| $\cos x$ | \mathbb{R} | $[-1, 1]$ |
| $\tan x$ | $\mathbb{R} - \{(2n+1)\pi/2, n \in \mathbb{Z}\}$ | \mathbb{R} |
| $\cot x$ | $\mathbb{R} - \{n\pi, n \in \mathbb{Z}\}$ | \mathbb{R} |
| $\sec x$ | $\mathbb{R} - \{(2n+1)\pi/2, n \in \mathbb{Z}\}$ | $(-\infty, -1] \cup [1, \infty)$ |
| $\operatorname{cosec} x$ | $\mathbb{R} - \{n\pi, n \in \mathbb{Z}\}$ | $(-\infty, -1] \cup [1, \infty)$ |

Trigonometric Ratios of negative angles: -

$$\begin{aligned} \textcircled{1} \sin(-\theta) &= \frac{-P}{H} = \frac{-P}{H} \\ &= -\left(\frac{P}{H}\right) \\ &= -\sin\theta \end{aligned}$$



$$\textcircled{2} \cos(-\theta) = \frac{P}{H} = \cos\theta$$

similarly

$$\textcircled{3} \tan(-\theta) = -\tan\theta$$

$$\textcircled{4} \cot(-\theta) = -\cot\theta$$

$$\textcircled{5} \sec(-\theta) = \sec\theta$$

$$\textcircled{6} \operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$$

NOTE :- \cos & \sec are even trigonometric functions. and others are odd functions.

$$\underline{\text{Q:-1}} \quad \tan(-840^\circ)$$

$$= -\tan 840^\circ = -\tan \{120^\circ\} \text{ (using co-terminal angles)}$$

$$= -\tan(90^\circ + 30^\circ)$$

$$= -[-\cot 30^\circ] = \cot 30^\circ = \sqrt{3}$$

Trigonometric functions of compound angles:-

$$\textcircled{1} \sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\textcircled{2} \sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\textcircled{3} \cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\textcircled{4} \cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\textcircled{5} \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\textcircled{6} \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\textcircled{7} \cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

$$\textcircled{8} \cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

Q:- find $\cos 15^\circ$

Solⁿ $\cos 15^\circ = \cos(45^\circ - 30^\circ)$

$$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

Trigonometric formulas of multiple angles:-

$$\textcircled{1} \sin 2A = 2 \sin A \cos A$$

$$\text{or } \frac{2 \tan A}{1 + \tan^2 A}$$

$$\textcircled{2} \cos 2A = \cos^2 A - \sin^2 A$$

$$\text{or } 2 \cos^2 A - 1$$

$$\text{or } 1 - 2 \sin^2 A$$

$$\text{or } \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\textcircled{3} \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\textcircled{4} \sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\textcircled{5} \cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\textcircled{6} \tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

Proof $\textcircled{1}$ $\sin 2A = \sin(A+A) = \sin A \cos A + \cos A \sin A$
 $= 2 \sin A \cos A$

$$\text{or } \sin 2A = \frac{2 \sin A \cos A}{\cos^2 A} \cdot \cos^2 A$$
$$= \frac{2 \tan A}{\sec^2 A} = \frac{2 \tan A}{1 + \tan^2 A}$$

Proof (2) $\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$$

$$= \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

Q:- Evaluate $\sin 18^\circ$

$$A = 18^\circ$$

$$\Rightarrow 5A = 90^\circ$$

$$\Rightarrow 2A + 3A = 90^\circ$$

$$\Rightarrow 2A = 90^\circ - 3A$$

$$\Rightarrow \sin 2A = \sin(90^\circ - 3A) = \cos 3A$$

$$\Rightarrow 2 \sin A \cos A = 4 \cos^3 A - 3 \cos A$$

$$\Rightarrow 2 \sin A = 4 \cos^2 A - 3 = 4(1 - \sin^2 A) - 3$$

$$= 4 - 4 \sin^2 A - 3$$

$$\Rightarrow 4 \sin^2 A + 2 \sin A - 1 = 0$$

$$\sin A = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{\pm \sqrt{5}-1}{4}$$

as 18° lies on 1st Quadrant so ~~angle's~~ value will be positive.

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4}$$

Trigonometric Formulas of Sub Multiple Angle

$$\textcircled{1} \sin A = 2 \sin \frac{A}{2} \cdot \cos \frac{A}{2} \quad \text{or} \quad \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}$$

$$\textcircled{2} \cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$$

$$\text{or } 2 \cos^2 \frac{A}{2} - 1$$

$$\text{or } 1 - 2 \sin^2 \frac{A}{2}$$

$$\text{or } \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}$$

$$\textcircled{3} \tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$$

$$\textcircled{4} \sin^2 \frac{A}{2} + \cos^2 \frac{A}{2} = 1$$

Q:- Prove that $\cot 7\frac{1}{2}^\circ = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2$

Proof $\cot \theta = \frac{1 + \cos \theta}{\sin \theta}$

$$\Rightarrow \cot \frac{\theta}{2} = \frac{1 + \cos \theta}{\sin \theta}$$

put $\theta = 15^\circ$

$$\Rightarrow \cot 7\frac{1}{2}^\circ = \frac{1 + \cos 15^\circ}{\sin 15^\circ}$$

$$= \frac{1 + \frac{\sqrt{3} + 1}{2\sqrt{2}}}{\frac{\sqrt{3} - 1}{2\sqrt{2}}}$$

$$= \frac{2\sqrt{2} + \sqrt{3} + 1}{\sqrt{3} - 1}$$

$$= \frac{(2\sqrt{2} + \sqrt{3} + 1)(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)} = \frac{2\sqrt{6} + 2\sqrt{3} + 2\sqrt{2} + 4}{2}$$

$$= \sqrt{6} + \sqrt{3} + \sqrt{2} + 2 \text{ (proved).}$$

Some Special Formulas:

$$\textcircled{1} \sin(A+B) + \sin(A-B) = 2 \sin A \cdot \cos B$$

$$\textcircled{2} \sin(A+B) - \sin(A-B) = 2 \cos A \cdot \sin B$$

$$\textcircled{3} \cos(A+B) + \cos(A-B) = 2 \cos A \cdot \cos B$$

$$\textcircled{4} \cos(A+B) - \cos(A-B) = -2 \sin A \cdot \sin B$$

Put $A = \frac{C+D}{2}$, $B = \frac{C-D}{2}$; Then $A+B = C$
 $A-B = D$

$$\textcircled{5} \sin C + \sin D = 2 \sin \frac{C+D}{2} \cdot \cos \frac{C-D}{2}$$

$$\textcircled{6} \sin C - \sin D = 2 \cos \frac{C+D}{2} \cdot \sin \frac{C-D}{2}$$

$$\textcircled{7} \cos C + \cos D = 2 \cos \frac{C+D}{2} \cdot \cos \frac{C-D}{2}$$

$$\textcircled{8} \cos C - \cos D = -2 \sin \frac{C+D}{2} \cdot \sin \frac{C-D}{2}$$

Two-Dimensional Geometry :- (Co-ordinate Geometry)

Introduction :-

→ Co-ordinate Geometry is a link between the geometry and algebra, in which the geometrical problems are solved through algebra using curves and lines.

→ It is a part of geometry, where the position of points is described using an ordered pair of numbers on the plane.

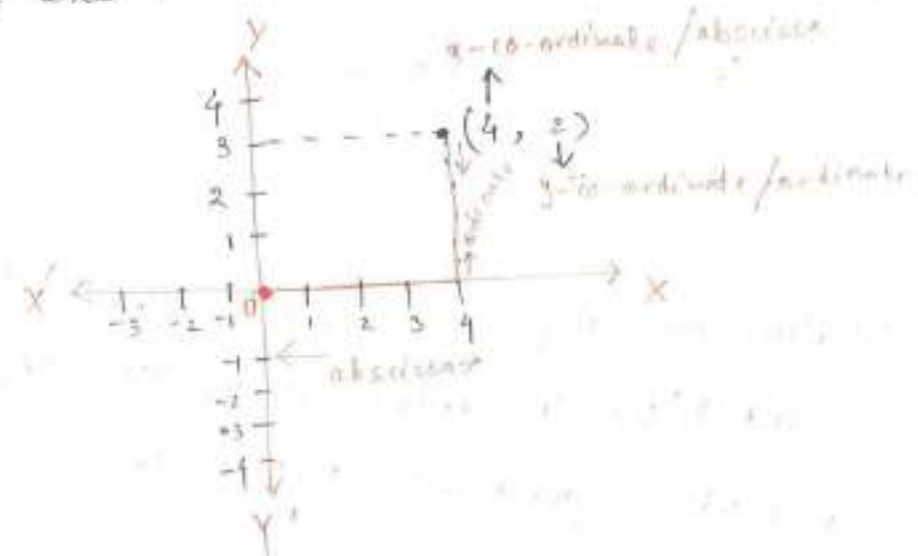
Application:- using co-ordinate geometry, it is possible to find the distance between two pts, to calculate area of a triangle in co-ordinate plane.

Co-ordinate :-

Co-ordinates are a set of values which helps to show the exact position of a pt. in the co-ordinate plane.

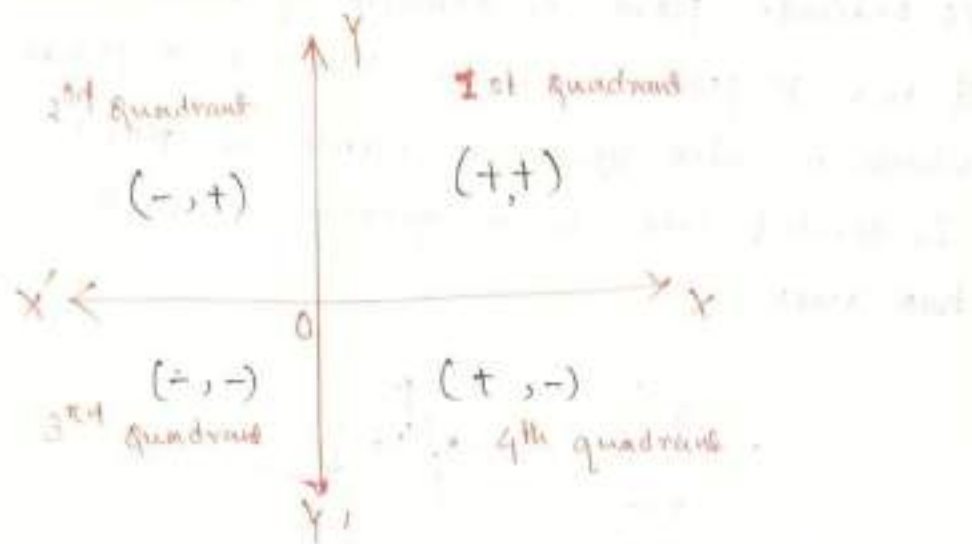
Co-ordinate plane :-

Co-ordinate plane is formed by intersection of two perpendicular lines X-axis & Y-axis, which is also known as Cartesian plane. It is divided into four quadrants by the two axes.



- * The point at which the axes intersect is known as the origin.
- * The location of any pt. on a plane is expressed by a pair of values (x, y), known as the co-ordinates.
- * The horizontal line (X-axis) & vertical line (Y-axis) are known as co-ordinate axes.

Quadrants:



- Hence the regions xOy' , yOx' , $x'Oy'$ and $y'Ox$ are known as the 1st, 2nd, 3rd and 4th quadrant respectively.
- Signs for a point in different quadrants are given as follows:

- 1st quadrant $\rightarrow (+x, +y)$
- 2nd quadrant $\rightarrow (-x, +y)$
- 3rd quadrant $\rightarrow (-x, -y)$
- 4th quadrant $\rightarrow (+x, -y)$

Co-ordinate Geometry Formulas &

Theorems:

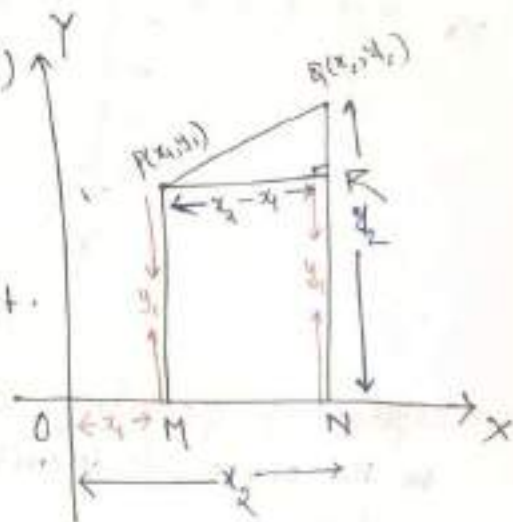
Measurement: - The distance between two pts $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

→ Hence let $P(x_1, y_1)$ & $Q(x_2, y_2)$ be two pts.

→ Let's draw perpendiculars PM and QN from the pts. P and Q on x -axis.

→ Also draw $PR \perp QN$.



$$\text{Then } OM = x_1 \quad ON = x_2 \\ PM = y_1 \quad QN = y_2$$

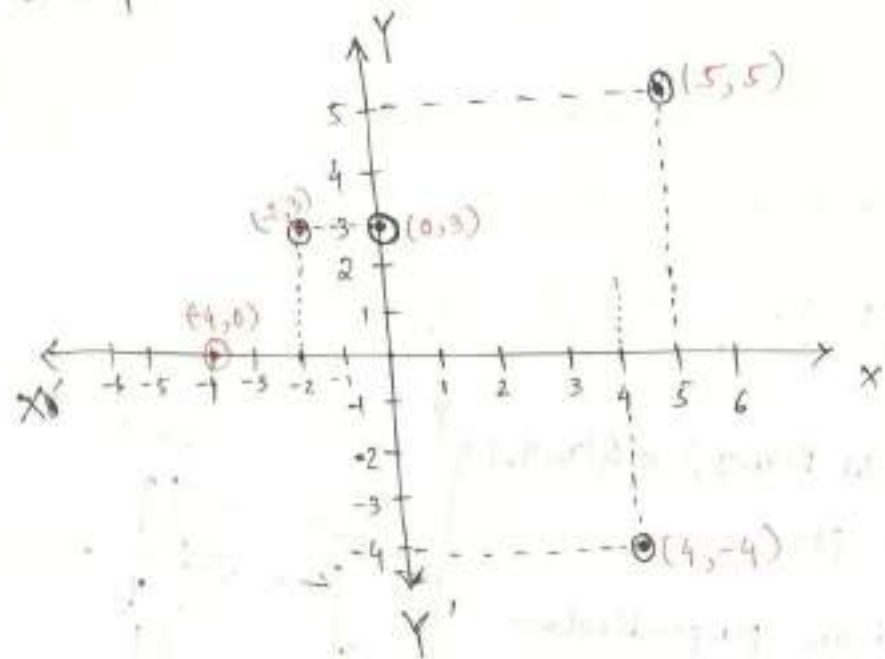
$$\text{Then } PR = MN = ON - OM = x_2 - x_1$$

$$\text{Again } QR = QN - RN = y_2 - y_1$$

Now consider the right angled triangle PQR .

Plotting of pts on cartesian plane:-

Let's plot $(-2, 3)$, $(4, -4)$, $(5, 5)$, $(0, 3)$, $(-4, 0)$



* System of geometry where the position of pts on the plane is described by using ordered pair of numbers.

* The plane where the pts are placed on, is known as co-ordinate plane.

It has two dimensions.

* A pt's location on a plane is given by two numbers, 1st tells where it is on x-axis, & second where it is on y-axis. & together they define a single & unique position on the plane.

By Pythagoras Theorem,
we have $(PQ)^2 = (PR)^2 + (QR)^2$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\Rightarrow \boxed{PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

Q.1:- Find the distance between the pts $(2, -4)$, $(5, 6)$.

Solution:- Let $P(2, -4)$ and $Q(5, 6)$ are two given pts.

$$P(2, -4), \quad x_1 = 2, \quad y_1 = -4$$

$$Q(5, 6), \quad x_2 = 5, \quad y_2 = 6$$

$$\text{Then } PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(5-2)^2 + (6-(-4))^2}$$

$$= \sqrt{(3)^2 + (10)^2}$$

$$= \sqrt{9+100} = \sqrt{109}$$

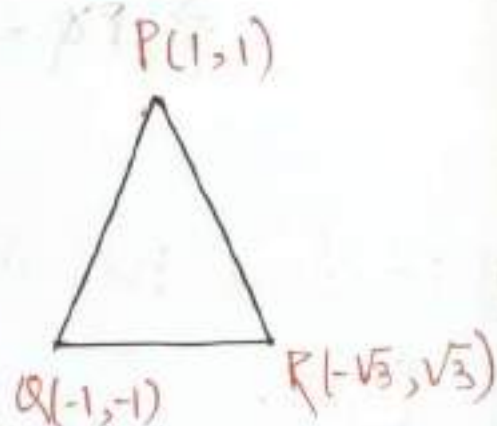
Q: 2 Show that the pts $(1, 1)$, $(-1, -1)$ and $(-\sqrt{3}, \sqrt{3})$ are the vertices of an equilateral triangle.

$$\text{Let } P(1, 1) \Rightarrow x_1 = 1, y_1 = 1$$

$$Q(-1, -1) \Rightarrow x_2 = -1, y_2 = -1$$

$$R(-\sqrt{3}, \sqrt{3}) \Rightarrow x_3 = -\sqrt{3}, y_3 = \sqrt{3}$$

are three given pts.



$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(-1 - 1)^2 + (-1 - 1)^2}$$

$$= \sqrt{(-2)^2 + (-2)^2} = \sqrt{4 + 4} = \sqrt{8}$$

$$QR = \sqrt{(-\sqrt{3} - (-1))^2 + (\sqrt{3} - (-1))^2}$$

$$= \sqrt{(-\sqrt{3} + 1)^2 + (\sqrt{3} + 1)^2}$$

$$= \sqrt{(1)^2 + (\sqrt{3})^2 + 2(\sqrt{3})(1) + (\sqrt{3} + 1)^2 + 2(\sqrt{3})(1)}$$

$$= \sqrt{1 + 3 + 3 + 1} = \sqrt{8}$$

$$= \sqrt{8}$$

$$\begin{aligned} PR &= \sqrt{(-\sqrt{3}-1)^2 + (\sqrt{3}-1)^2} \\ &= \sqrt{(-\sqrt{3})^2 + (-1)^2 + 2(-\sqrt{3})(-1) + (\sqrt{3})^2 + (1)^2 + 2(\sqrt{3})(-1)} \\ &= \sqrt{3+1+3+1} \\ &= \sqrt{8} \end{aligned}$$

Hence $PQ = QR = PR$

\Rightarrow PQR is an equilateral triangle.

Theorem-2 (Section formula)

Section formula helps in finding the co-ordinate of a pt. which divides a line segment in some ratio let $m:n$.

⇒ If $m=n$, then the pt. is the midpoint.

Internal division with Section formula

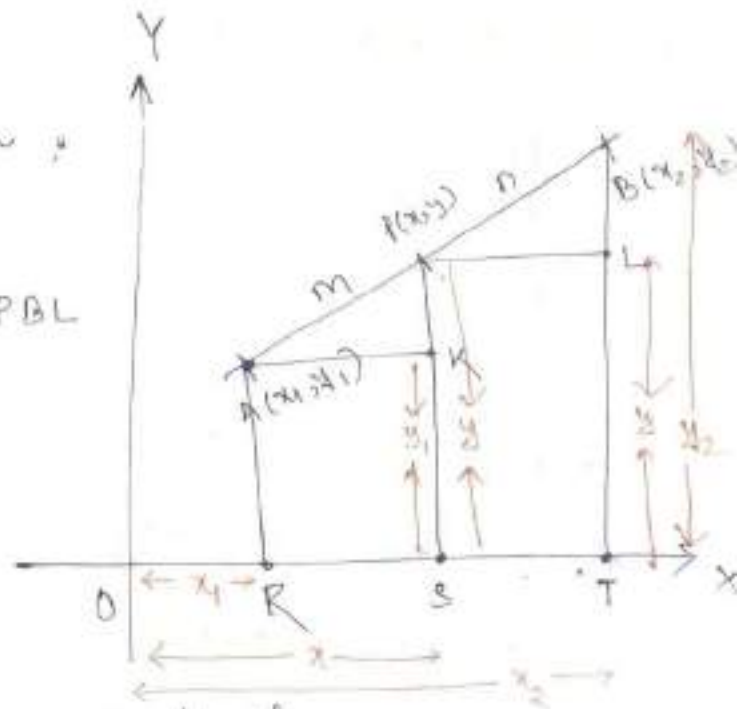
Let $P(x, y)$ be a pt. which lies on a line-segment \overline{AB} and satisfies $AP:PB = m:n$ then we can say P divides the line \overline{AB} internally in the ratio $m:n$.

Then co-ordinates of P will be

$$= \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right)$$

In the given figure

$\triangle APK$ & $\triangle PBL$ are similar.



⇒ Sides are proportional.

$$\text{i.e. } \frac{AP}{PB} = \frac{AK}{PL} = \frac{PK}{BL}$$

consider first two ratios.

$$\frac{AP}{PB} = \frac{AK}{PL}$$

$$\Rightarrow \frac{m}{n} = \frac{x-x_1}{x_2-x}$$

$$\Rightarrow m(x_2-x) = n(x-x_1)$$

$$\Rightarrow mx_2 - mx = nx - nx_1$$

$$\Rightarrow mx_2 + nx_1 = mx + nx$$

$$\Rightarrow mx_2 + nx_1 = x(m+n)$$

$$\Rightarrow x = \frac{mx_2 + nx_1}{m+n}$$

Consider 1st and 3rd ratios.

$$\frac{AP}{PB} = \frac{PK}{BL}$$

$$\Rightarrow \frac{m}{n} = \frac{y - y_1}{y_2 - y}$$

$$\Rightarrow my_2 - my = ny - ny_1$$

$$\Rightarrow my_2 + ny_1 = ny + my$$

$$\Rightarrow my_2 + ny_1 = (m+n)y$$

$$\Rightarrow y = \frac{my_2 + ny_1}{m+n}$$

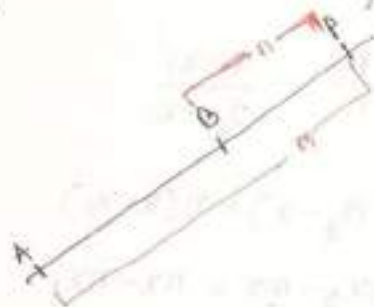
External Division with Section Formula

Let $P(x, y)$ be a pt. lies on the ~~extension~~ ^{extension} of the line segment AB and satisfies $AP:BP = m:n$

\Rightarrow P divides the line segment externally in the ratio $m:n$

$$\text{Then } x = \frac{mx_2 - nx_1}{m-n}$$

$$y = \frac{my_2 - ny_1}{m-n}$$

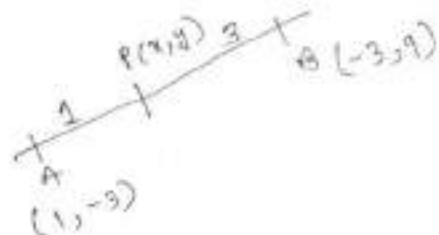


Q:-1 :- Find the co-ordinates of the pt. which divides the line segment $(1, -3)$ and $(-3, 9)$ in the ratio $1:3$

Solⁿ

$$\text{Let } A(1, -3) \parallel \begin{cases} x_1 = 1 \\ y_1 = -3 \end{cases}$$

$$\text{and } B(-3, 9) \parallel \begin{cases} x_2 = -3 \\ y_2 = 9 \end{cases}$$



~~It~~ makes the line segment AB.

and $P(x, y)$ divides AB in the ratio $1:3$

$$\text{So } m = 1$$

$$n = 3$$

Then co-ordinates of P is obtained by

$$= \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right)$$

$$= \left(\frac{1(-3) + 3(1)}{1+3}, \frac{1(9) + 3(-3)}{1+3} \right)$$

$$= \left(\frac{0}{4}, \frac{0}{4} \right) = (0, 0)$$

Q.2 Find the co-ordinate of the pt. which divides the line segment joining $(1, -3)$ & $(-3, 9)$ in the ratio 1:3.

Solution: Let $P(x, y)$ be the pt which divides the line segment joining $(1, -3) \parallel \begin{cases} x_1 = 1 \\ y_1 = -3 \end{cases}$

and $(-3, 9) \parallel \begin{cases} x_2 = -3 \\ y_2 = 9 \end{cases}$

in the ratio 1:3

$$\Rightarrow m = 1$$

$$n = 3$$

$$\text{Then } x = \frac{mx_2 + nx_1}{m+n} = \frac{1(-3) + 3(1)}{1+3} = \frac{-3+3}{4} = \frac{0}{4} = 0$$

$$y = \frac{my_2 + ny_1}{m+n} = \frac{1(9) + 3(-3)}{1+3} = \frac{9-9}{4} = \frac{0}{4} = 0$$

So the required co-ordinate is $(0, 0)$.

Q.3 Find the ratio in which the pt $(3, -2)$ divides the line segment joining pts $(1, 4)$ & $(-3, 16)$.

Solution: - Let $P(3, -2) \parallel \begin{cases} x = 3 \\ y = -2 \end{cases}$

divides the line segment joining $(1, 4) \parallel \begin{cases} x_1 = 1 \\ y_1 = 4 \end{cases}$

and $(-3, 16) \parallel \begin{cases} x_2 = -3 \\ y_2 = 16 \end{cases}$

in the ratio $m:n$

$$\text{Then } x = \frac{mx_2 + nx_1}{m+n}$$

$$\Rightarrow 3 = \frac{m(-3) + n(1)}{m+n}$$

$$\Rightarrow 3(m+n) = -3m+n$$

$$\Rightarrow 3m+3n = -3m+n$$

$$\Rightarrow 6m = -2n$$

$$\Rightarrow \frac{m}{n} = \frac{-2}{6} = \frac{-1}{3}$$

$\therefore P(3, -2)$ divides the line segment externally in the ratio 1:3.

2. External

Q.4 find the co-ordinate of the pt. which divides the line segment joining $(2, -1)$ & $(-3, 4)$ in the ratio 2:3 externally.

Solution:- let $P(x, y)$ be the pt. which divides the line segment joining $(2, -1)$ & $(-3, 4)$ in the ratio 2:3 externally.

$$\text{Then } x = \frac{mx_2 + nx_1}{m-n} = \frac{2(-3) + 3(2)}{2-3} = \frac{-6+6}{-1} = \frac{-12}{-1} = 12$$

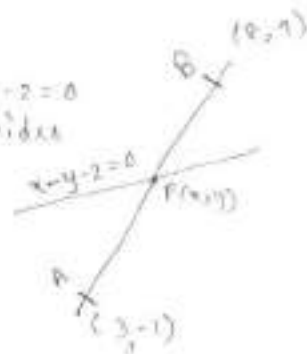
$$y = \frac{my_2 + ny_1}{m-n} = \frac{2(4) + 3(-1)}{2-3} = \frac{8-3}{-1} = \frac{5}{-1} = -5$$

So the required co-ordinate of pt. 'P' is $(12, -5)$.

Q.5 find the ratio in which the line $x-y-2=0$ cuts the line segment joining $(5, -1)$ & $(8, 9)$.

Solution

Let $P(x, y)$ be a pt. which divides the line segment joining $A(5, -1)$ and $B(8, 9)$ in the ratio $m:n$.



$$\text{Then } x = \frac{mx_2 + nx_1}{m+n} \quad \& \quad y = \frac{my_2 + ny_1}{m+n}$$

$$= \frac{8m + 5n}{m+n} \quad = \frac{9n - 1}{m+n}$$

As $P\left(\frac{8m+5n}{m+n}, \frac{9n-1}{m+n}\right)$ be a pt. on $x-y-2=0$

$$\Rightarrow \frac{8m+5n}{m+n} - \frac{9n-1}{m+n} - 2 = 0$$

$$\Rightarrow \frac{8m+5n-9n+1-2(m+n)}{m+n} = 0$$

$$\Rightarrow \frac{8m+5n-9n+1-2m-2n}{m+n} = 0$$


$$\Rightarrow -3m + 2n = 0 \quad \Rightarrow 3m = 2n$$

$$\Rightarrow \frac{m}{n} = \frac{2}{3}$$

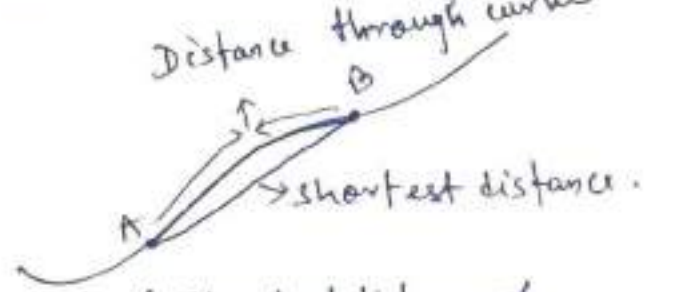
STRAIGHT LINE

Definition :- In a straight line if we take any two points then the shortest distance between the points is equal to the distance between the pts through the curve.

Hence
Shortest distance
= Distance
through curve (Straight line)

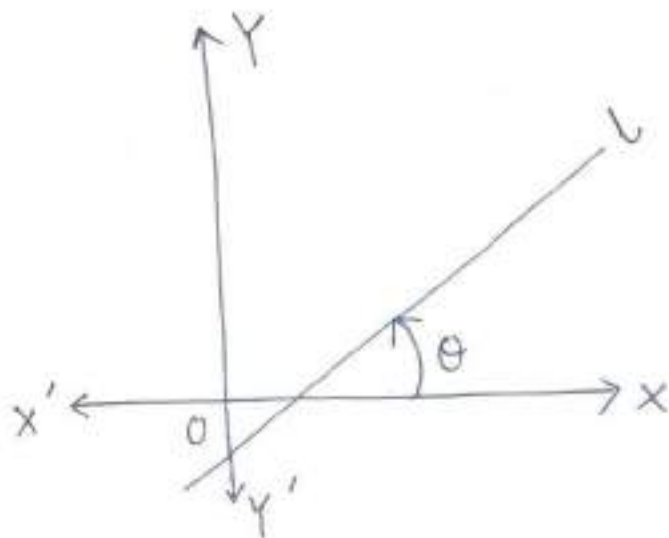


Distance through curve
shortest distance.
(Shortest distance \neq
distance through curve)



Inclination of a straight line :-

Inclination of a line is the angle made by the line with the positive X-axis and measured in anti clockwise direction.



Here θ is the inclination of the st. line 'l'.

* Inclination of X-axis or any line parallel to X-axis is 0°

* Inclination of Y-axis or line \parallel to Y-axis is 90° .

Slope of a line :-

① Slope of a line which makes an angle θ with positive x-axis is $\tan \theta$ and denoted by the letter 'm'.

i.e. if inclination of a line is ' θ '
 then its slope is $m = \tan \theta$

* Slope of a line \parallel to x-axis or x-axis is
 $m = \tan 0^\circ = 0$ (or slope of horizontal lines is 0)

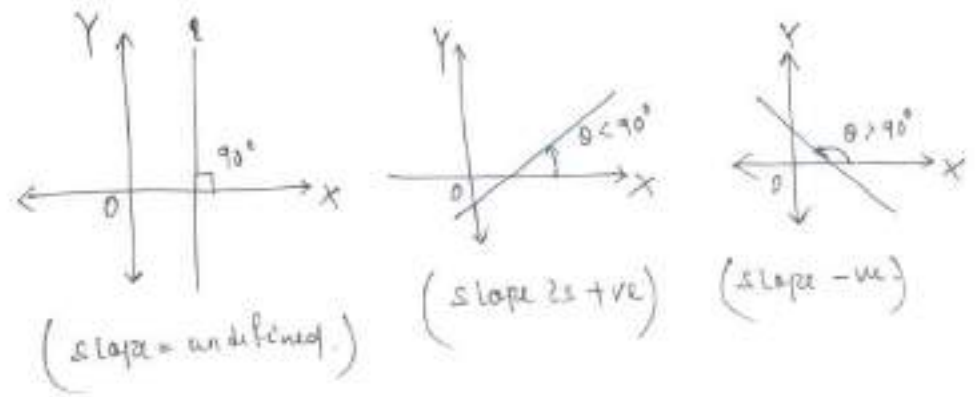
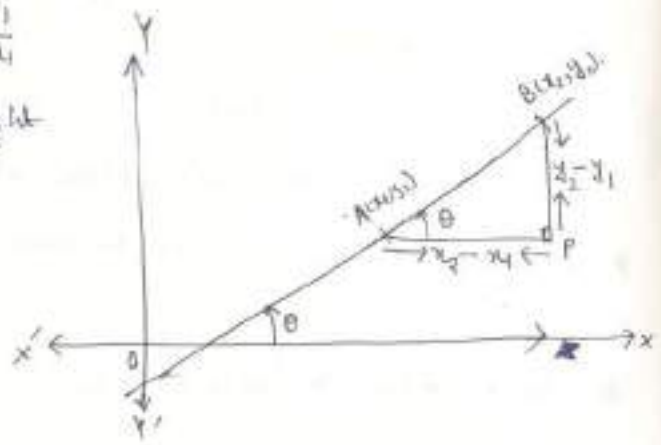
* Slope of a line \parallel to y-axis or y-axis is
 $m = \tan 90^\circ = \text{undefined}$
 (or slope of vertical lines is 0)

② Slope of line passing through two fixed pts A(x₁, y₁) and B(x₂, y₂)

slope = $\tan \theta = \frac{P}{B} = \frac{y_2 - y_1}{x_2 - x_1}$

by considering the right angle ΔABP

$m = \frac{y_2 - y_1}{x_2 - x_1}$



Angle between two lines :-

Let l_1 and l_2 be two lines with slope m_1 & m_2 respectively.

Then here ' θ ' will be the angle between the two lines.



\Rightarrow Here inclination of l_1 & l_2 be θ_1 & θ_2 respectively (let).

Exterior angle = sum of two interior opposite angles

Then $\theta_2 = \theta + \theta_1$
 $\Rightarrow \theta = \theta_2 - \theta_1$
 $\Rightarrow \tan \theta = \tan (\theta_2 - \theta_1)$
 $= \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_2 \tan \theta_1}$

$$\Rightarrow \tan \theta = \frac{m_2 - m_1}{1 + m_2 m_1} \quad \left(\begin{array}{l} \text{as } m_1 = \tan \theta_1 \\ m_2 = \tan \theta_2 \end{array} \right)$$

$$\text{or } \tan \theta = \left| \frac{m_2 - m_1}{1 + m_2 m_1} \right| \quad (\text{as } \theta \text{ is an acute angle})$$

$$\text{or } \tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\text{or } \theta = \tan^{-1} \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

NOTE 1 condition of parallelism of two lines.

Two lines are parallel

\Rightarrow angle between them is zero.

$$\Rightarrow \tan 0 = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\Rightarrow 0 = m_1 - m_2 \Rightarrow \boxed{m_1 = m_2}$$

NOTE 2 condition of perpendicularity

Two lines are perpendicular.

\Rightarrow angle between them is 90°

$$\Rightarrow \tan 90^\circ = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\Rightarrow \infty = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\Rightarrow 0 = \frac{1 + m_1 m_2}{m_1 - m_2}$$

$$\Rightarrow m_1 m_2 + 1 = 0$$

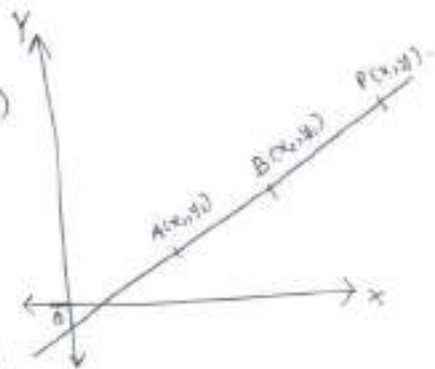
$$\Rightarrow \boxed{m_1 m_2 = -1}$$

EQUATION OF STRAIGHT LINE

① Equation of a line in two pt. form :-

Let L be a line passing through $A(x_1, y_1)$ and $B(x_2, y_2)$

& let's take $P(x, y)$ be a general pt. on the line.



Then slope of AP = Slope of AB

$$\Rightarrow \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)}$$
 required eqⁿ of the line

② Equation of line in point-slope form :-

Let m be the slope of the line

and the line passing through (x_1, y_1)

then required eqⁿ is

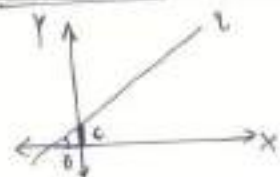
$$\boxed{y - y_1 = m(x - x_1)}$$

③ We have $y - y_1 = m(x - x_1)$ Slope-intercept form

$$\Rightarrow y - y_1 = mx - mx_1$$

$$\Rightarrow y = mx - mx_1 + y_1$$

$\Rightarrow \boxed{y = mx + c}$ also the eqⁿ of straight line
where coefficient of x is slope of line.



\Rightarrow So Eqⁿ of straight line is a linear eqⁿ in x & y .
i.e. $mx - y + c = 0$

④ General Eqⁿ of straight line

Any linear eqⁿ in x and y is the eqⁿ of a straight line.

i.e. $\boxed{ax + by + c = 0}$ is the required eqⁿ.

NOTE 1:-

$$by = -ax - c$$

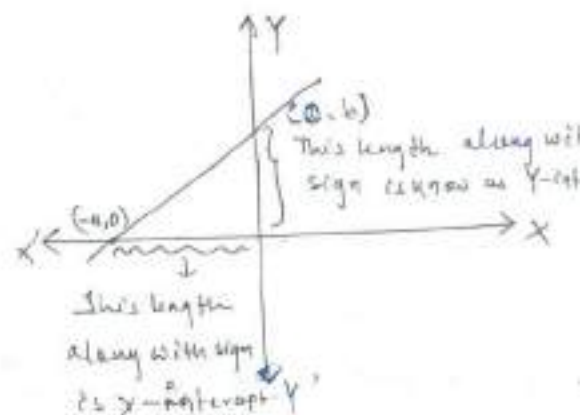
$$y = \left(-\frac{a}{b}\right)x + \left(-\frac{c}{b}\right)$$

which is in slope-intercept form.

$$\text{Then } \boxed{\text{slope} = -\frac{a}{b}}$$

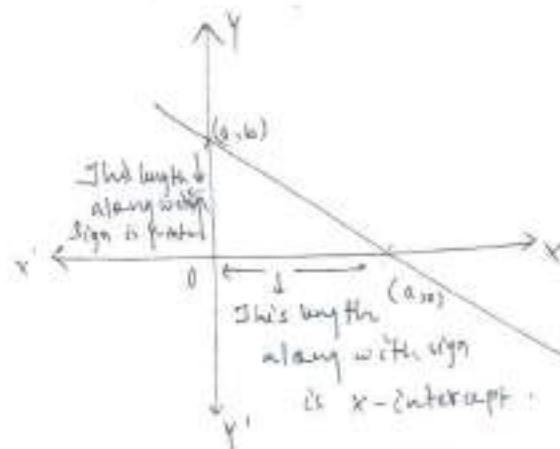
$$\boxed{y\text{-intercept} = -\frac{c}{b}}$$

⑤ Intercept form



X-intercept = $-a$

Y-intercept = b

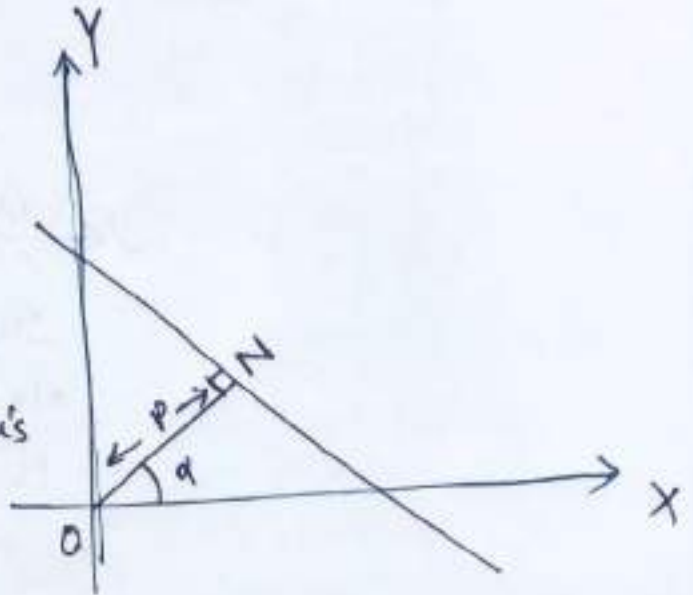


The eqⁿ of the line $\boxed{\frac{x}{a} + \frac{y}{b} = 1}$

(Using two point form)

⑥ Normal/perpendicular form :-

Let p be the length of perpendicular from the origin to a given line and α be the angle made by this perpendicular line with the x -axis



then eqⁿ of the line $x \cos \alpha + y \sin \alpha = p$

Reduction of General form to standard form

① Reduction of General form to slope intercept form.

$$\text{General form} \Rightarrow ax + by + c = 0$$

$$\Rightarrow by = -ax - c$$

$$\Rightarrow y = \left(-\frac{a}{b}\right)x + \left(-\frac{c}{b}\right)$$

So here

| |
|---------------------------------|
| slope = $-\frac{a}{b}$ |
| y -intercept = $-\frac{c}{b}$ |

② Intercept form

General form is

$$ax + by + c = 0$$

$$\Rightarrow ax + by = -c$$

$$\Rightarrow \frac{ax}{-c} + \frac{by}{-c} = 1$$

$$\Rightarrow \frac{x}{\left(-\frac{c}{a}\right)} + \frac{y}{\left(-\frac{c}{b}\right)} = 1$$

So here

| |
|---------------------------------|
| x -intercept = $-\frac{c}{a}$ |
| y -intercept = $-\frac{c}{b}$ |

* Another condition of parallelism of two line in general form

Two lines are \parallel

$$\text{then } m_1 = m_2$$

$$\Rightarrow \frac{-a_1}{b_1} = \frac{-a_2}{b_2}$$

$$\Rightarrow \boxed{\frac{a_1}{b_1} = \frac{a_2}{b_2}}$$

* Another condition of perpendicularity

$$\text{Let } a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

are two perpendicular

lines then $m_1 m_2 = -1$

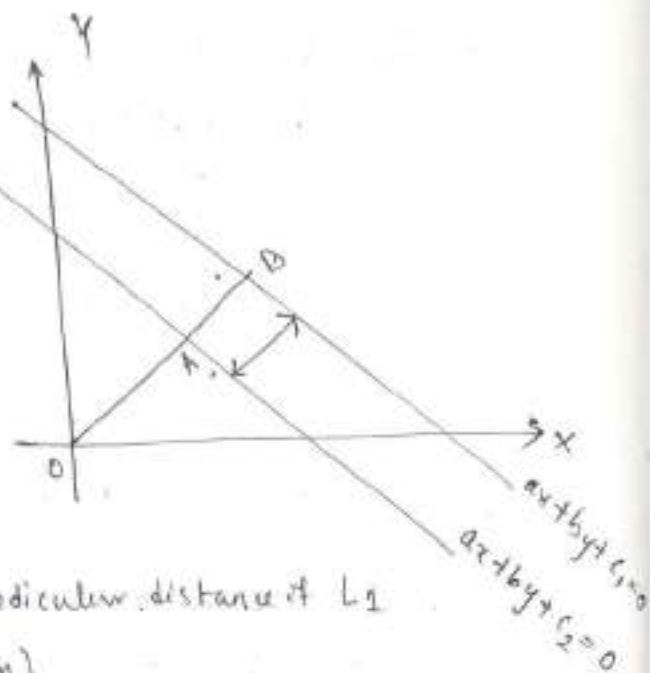
$$\Rightarrow \left(\frac{-a_1}{b_1}\right) \left(\frac{-a_2}{b_2}\right) = -1$$

$$\Rightarrow a_1 a_2 = -b_1 b_2$$

$$\Rightarrow \boxed{a_1 a_2 + b_1 b_2 = 0}$$

Distance between two parallel lines :-

Distance = shortest / perpendicular distance between two lines.



Hence OB (Perpendicular distance of L_1 from origin)

$$OB = \left| \frac{c_1}{\sqrt{a^2+b^2}} \right|$$

OA = Perpendicular distance of L_2 from origin.

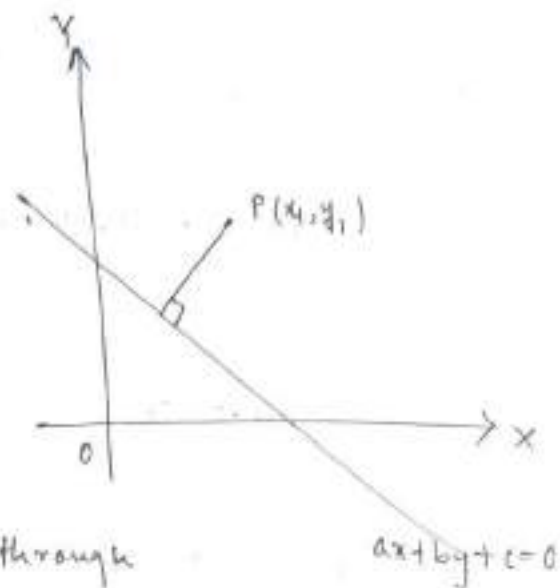
$$OA = \left| \frac{c_2}{\sqrt{a^2+b^2}} \right|$$

$$AB = OB - OA = \left| \frac{c_1 - c_2}{\sqrt{a^2+b^2}} \right|$$

So
$$D = \left| \frac{c_1 - c_2}{\sqrt{a^2+b^2}} \right|$$

So when using this formula (coefficient of x and y must be same in both lines.)

Distance between a Pt. (x_1, y_1) and a line $ax+by+c=0$
(Perpendicular Distance)



Eqⁿ of the line passing through pt $P(x_1, y_1)$ and parallel to $ax+by+c=0$

Then slope will be same as $-a/b$

Then using slope-intercept form.

Eqⁿ of the required line.

$$y - y_1 = -\frac{a}{b}(x - x_1)$$

$$\Rightarrow ax + by - ax_1 - by_1 = 0$$

$$\Rightarrow ax + by - (ax_1 + by_1) = 0$$

Then Distance between the lines.

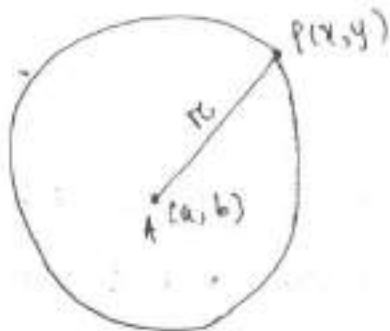
$$D = \left| \frac{c - f(ax_1 + by_1)}{\sqrt{a^2+b^2}} \right| = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}} \right|$$

CIRCLE

Definition:- Circle is a locus of a point 'P' which moves in such a way that its distance from a fixed point is always constant.

where fixed pt (a, b)
is centre.

$r = \text{radius} = \text{Distance AP}$



Standard Equation of a circle:-

Eqⁿ of the circle whose centre (a, b) and radius is 'r'

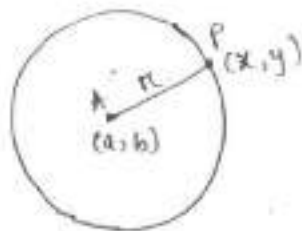
Distance of AP

$$r = \sqrt{(x-a)^2 + (y-b)^2}$$

Squaring both sides.

$$\Rightarrow r^2 = (x-a)^2 + (y-b)^2$$

which is the required eqⁿ of the circle.



General Eqⁿ of the circle:-

Standard form.

$$(x-a)^2 + (y-b)^2 = r^2$$

$$\Rightarrow x^2 + 2(-a)x + a^2 + y^2 - 2by + b^2 = r^2$$

$$\Rightarrow x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$$

$$\text{let } -a = g$$

$$-b = f$$

$$a^2 + b^2 - r^2 = c$$

$$\Rightarrow x^2 + y^2 + 2gx + 2fy + c = 0$$

required eqⁿ of circle.

$$\text{Then centre } (a, b) = (-g, -f) = \text{centre}$$

radius = r

$$\text{we have } a^2 + b^2 - r^2 = c$$

$$\Rightarrow r^2 = a^2 + b^2 - c$$

$$= (-g)^2 + (-f)^2 - c$$

$$\Rightarrow r = \sqrt{g^2 + f^2 - c}$$

NOTE :- $r = \sqrt{g^2 + f^2 - c}$

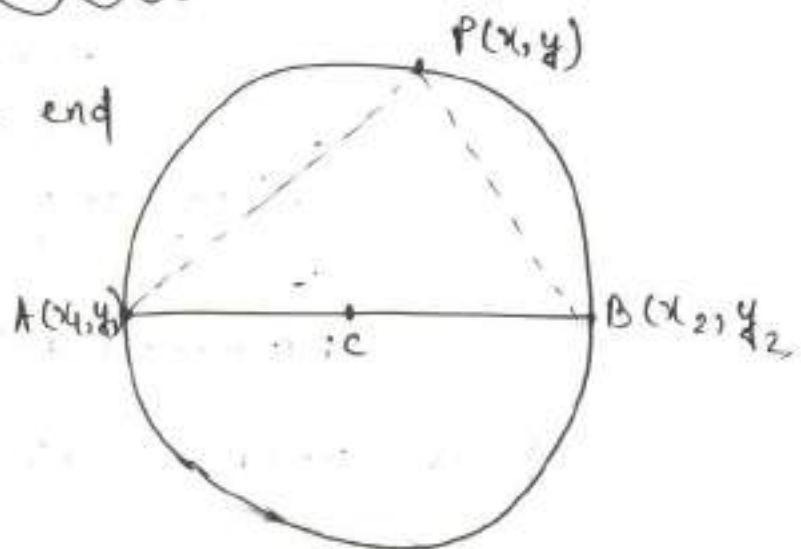
(i) if $g^2 + f^2 - c > 0 \Rightarrow$ real circle.

(ii) if $g^2 + f^2 - c < 0 \Rightarrow$ imaginary circle.

(iii) $g^2 + f^2 - c = 0 \Rightarrow$ point circle. of radius '0'.

Diametrical Eqⁿ of a circle :-

Eqⁿ of a circle whose end pts of diameter are (x_1, y_1) and (x_2, y_2)



Here $AP \perp BP$

$$\Rightarrow \text{slope of } AP \cdot \text{slope of } BP = -1$$

$$\Rightarrow \frac{y - y_1}{x - x_1} \cdot \frac{y - y_2}{x - x_2} = -1$$

$$\Rightarrow (y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$$

$$\Rightarrow \boxed{(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0}$$

Co-ordinate Geometry in three dimensions +

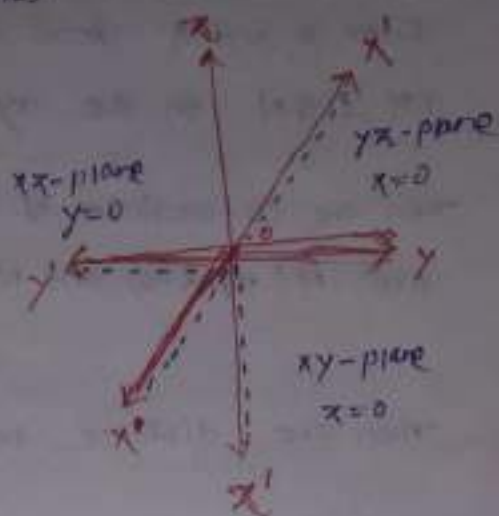
We take three perpendicular lines as axes.

O, the point of intersection is called origin.

$x'Ox$ is called x -axis.

$y'Oy$ is called y -axis.

$z'Oz$ is called z -axis.



The three lines taken together are called rectangular co-ordinate axes.

x -axis written as $(x, 0, 0)$.

y -axis written as $(0, y, 0)$.

z -axis written as $(0, 0, z)$.

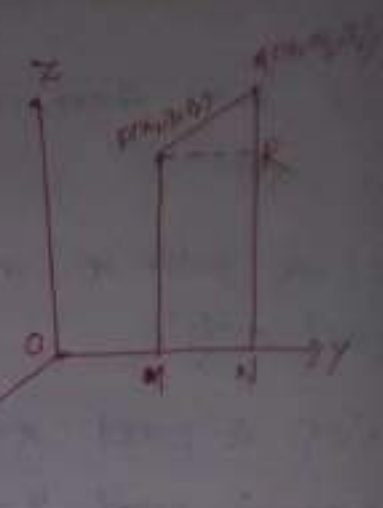
Distance Formula, Division Formula +

Theorem-1 (Distance Formula)

Prove that the distance betⁿ the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by.

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Proof Let O be origin and Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the given point. From P and Q draw perpendiculars PM & QN on the xy -plane.



then the co-ordinates of M and N are $M(x_1, y_1, 0)$ and $N(x_2, y_2, 0)$

then the distance betⁿ $MN = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (0 - 0)^2}$

$$\Rightarrow MN^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Now, from P draw $PR \perp QN$.

then PR is parallel and equal to MN .

Now, in right angle triangle PRQ , we have,

$$\begin{aligned} PQ^2 &= PR^2 + RQ^2 \\ &= MN^2 + (QN - RN)^2 \\ &= MN^2 + (QN - PM)^2 \end{aligned}$$

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad [\because PM = z_1 \text{ \& } QN = z_2]$$

Corollary - The distance of the point $P(x, y, z)$ from the origin $O(0, 0, 0)$ is

$$\begin{aligned} OP &= \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2} \\ &= \sqrt{x^2 + y^2 + z^2} \end{aligned}$$

Ex-2 Find the distance betⁿ two points (2, 3, 5) and (4, 3, 1)

Distance betⁿ two points (2, 3, 5) and (4, 3, 1) is

$$= \sqrt{(4-2)^2 + (3-3)^2 + (1-5)^2}$$

$$= \sqrt{2^2 + 0^2 + (-4)^2}$$

$$= \sqrt{4+16} = \sqrt{20} = 2\sqrt{5} \quad \text{Ans}$$

Ex-3 Find the value of x if distance betⁿ two points (x, -3, 4) and (3, -5, 4) is 5.

Solⁿ Given points, (x, -3, 4) and (3, -5, 4)

The distance betⁿ these points = 5

$$\sqrt{(x-3)^2 + (-3+5)^2 + (4-4)^2} = 5$$

$$\text{or, } (x-3)^2 + 4 + 0 = 25 \Rightarrow (x-3)^2 = 16$$

$$\text{or, } (x-3) = \pm 4 \quad \text{or, } x = 7, -1$$

Ex-3 Show that the points A(-2, -6, -7), B(4, -4, -5), C(7, -3, -4) are collinear.

Solⁿ We have

$$|AB| = \sqrt{(4+2)^2 + (-4+6)^2 + (-5+7)^2}$$

$$= \sqrt{6^2 + 2^2 + 2^2}$$

$$= \sqrt{36+4+4}$$

$$= \sqrt{44} = 2\sqrt{11}$$

$$|BC| = \sqrt{(7-4)^2 + (-3+4)^2 + (-4+5)^2}$$

$$= \sqrt{3^2 + 1^2 + 1^2}$$

$$= \sqrt{9+1+1}$$

$$= \sqrt{11}$$

$$|CA| = \sqrt{(-2-7)^2 + (-6+3)^2 + (-7+4)^2}$$

$$= \sqrt{81+9+9} = \sqrt{99} = 3\sqrt{11}$$

$$\therefore |AB| + |BC| = |CA|$$

Hence the points A, B, C are collinear.

Ex-4

Show that the points A(1, 2, 3), B(-1, -2, -1), C(2, 1, 2) and D(4, 7, 6) are the vertices of a parallelogram ABCD, but it is not a rectangle.

Solⁿ

We have,

$$AB = \sqrt{(-1-1)^2 + (-2-2)^2 + (-1-3)^2} = \sqrt{4+16+16} = 6$$

$$BC = \sqrt{(2+1)^2 + (1+2)^2 + (2+1)^2} = \sqrt{9+9+9} = \sqrt{27}$$

$$CD = \sqrt{(4-2)^2 + (7-3)^2 + (6-2)^2} = \sqrt{4+16+16} = 6$$

$$DA = \sqrt{(1-4)^2 + (2-7)^2 + (3-6)^2} = \sqrt{9+25+9} = \sqrt{43}$$

$\therefore AB = CD$ and $BC = DA$. Hence opposite sides are equal.

$\therefore ABCD$ is a parallelogram.

Again $AC = \sqrt{(2-1)^2 + (1-2)^2 + (2-3)^2}$

$$= \sqrt{1+1+1} = \sqrt{3}$$

$$BD = \sqrt{(4+1)^2 + (7+2)^2 + (6+1)^2}$$

$$= \sqrt{25+81+49} = \sqrt{155}$$

$\therefore AC \neq BD$ i.e. the diagonals are not equal.

Division Formula = (Ratio Formula)

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points and let $R(x, y, z)$ be a point on PQ dividing it in the ratio $m:n$ prove that:

$$\bar{x} = \frac{mx_2 + ny_1}{m+n}, \quad \bar{y} = \frac{my_2 + ny_1}{m+n}, \quad \bar{z} = \frac{mz_2 + nz_1}{m+n}$$

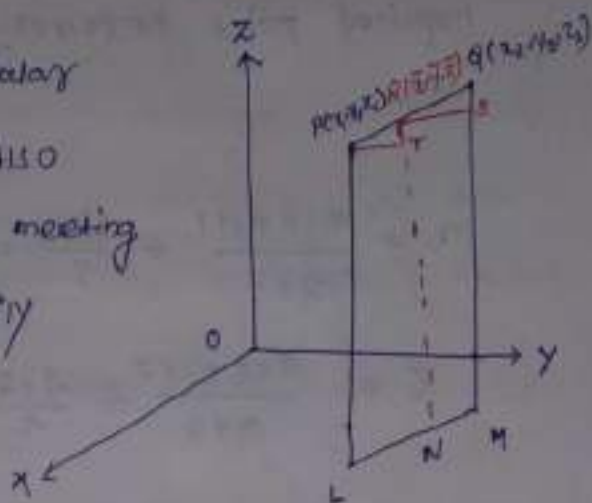
Let's draw P, Q, R perpendiculars

PL, QM and RN on xy -Plane. Also

draw $RS \perp QM$ and $PT \perp RN$ meeting

QM & RN at S and T respectively

PRT and QRS are similar
Triangles.



We have $\frac{RT}{QS} = \frac{PR}{RQ}$

$$\Rightarrow \frac{RN - TN}{QM - SM} = \frac{m}{n}$$

$$\Rightarrow \frac{\bar{x} - z_1}{z_2 - \bar{x}} = \frac{m}{n}$$

$$\Rightarrow n\bar{x} - nz_1 = mz_2 - m\bar{x}$$

$$\Rightarrow m\bar{x} + n\bar{x} = mz_2 + nz_1$$

$$\Rightarrow \bar{x}(m+n) = z_2(m+n)$$

$$\Rightarrow \bar{x} = \frac{mz_2 + nz_1}{m+n}$$

(Similarly $\bar{y} = \frac{my_2 + ny_1}{m+n}$)

For internal division -
Replacing n by $-n$.

$$\bar{x} = \frac{mx_2 - ny_1}{m-n}$$

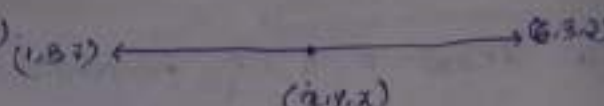
$$\bar{y} = \frac{my_2 - ny_1}{m-n}$$

$$\bar{z} = \frac{mz_2 - nz_1}{m-n}$$

Ex-1 Find the co-ordinates of a point which divides the points $(1, 3, 2)$, $(6, 3, 2)$ in the ratio $2:3$.

Sol

Let the co-ordinates of the required point be (x, y, z) .



$$x = \frac{2 \times 6 + 3 \times 1}{2 + 3} = \frac{12 + 3}{5} = 3, \quad y = \frac{2 \times 3 + 3 \times 3}{2 + 3} = \frac{6 + 9}{5} = 3$$

$$z = \frac{2 \times 2 + 3 \times 2}{2 + 3} = \frac{4 + 6}{5} = 2$$

Required point is $(3, 3, 2)$.

Sol
Ex-2

Find the ratio in which the line joining the points $(4, 4, -10)$ and $(-2, 2, 4)$ is divided by

(a) the yz -plane (b) $x + y + z = 3$.

Sol

The given points are $(4, 4, -10)$ and $(-2, 2, 4)$ is.

Let the ratio is $k:1$.

$$\text{Then } x = \frac{-2k + 4}{k + 1}$$

$$y = \frac{2k + 4}{k + 1}$$

$$z = \frac{4k - 10}{k + 1}$$

(a) If the plane lies in yz -plane then $x = 0$

$$\therefore \frac{-2k + 4}{k + 1} = 0 \text{ or } -2k + 4 = 0$$

$$\rightarrow -2k = -4$$

$$\rightarrow \boxed{k=2}$$

\therefore the required Ratio is 2:1.

$$(b) \quad x+y+z=3$$

$$\rightarrow \frac{-2k+4}{k+1} + \frac{2k+4}{k+1} + \frac{4k-10}{k+1} = 3$$

$$\rightarrow \frac{-2k+4 + 2k+4 + 4k-10}{k+1} = 3$$

$$\rightarrow \frac{4k-2}{k+1} = 3$$

$$\rightarrow 4k-2 = 3k+3$$

$$\rightarrow 4k-3k = 3+2$$

$$\rightarrow \boxed{k=5}$$

\therefore the required ratio is 5:1.

Ex-3 Find the ratio in which the line through $(2, 4, 5)$, $(3, 5, -4)$ is divided by xy -plane.

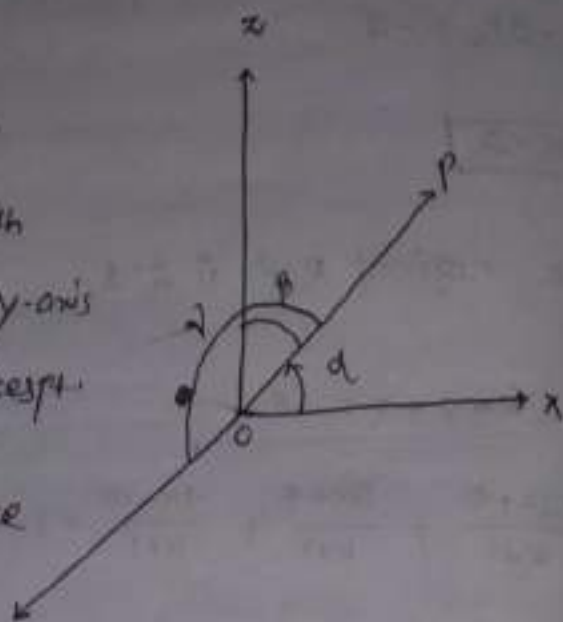
Ex-4 Find the ratio in which the line joining the points $(2, -3, 1)$, $(3, -4, -5)$ is divided by the plane $2x+y+z=7$.

Ex-5 Find the ratio in which the line segment joining the points $(4, 3, 2)$ & $(1, 2, -3)$ is divided by xy -plane.

Direction cosine

Let \vec{OP} be a st. line.

\vec{OP} makes angle α with
OX-axis and β with OY-axis
and γ with OZ-axis resp.



then $\cos \alpha$, $\cos \beta$ and $\cos \gamma$ are

called the direction
cosines (d.c.'s) of the line.

The (d.c.'s) of the line are denoted by l, m, n .

$$\therefore l = \cos \alpha, m = \cos \beta, n = \cos \gamma.$$

In particular the d.c.'s of x-axis are $1, 0, 0$.

Similarly the d.c.'s of y-axis and z-axis are
 $0, 1, 0$ and $0, 0, 1$ respectively.

Note:

$$l^2 + m^2 + n^2 = 1$$

Direction Ratios

The numbers a, b, c are called d.r.'s.

They are written in the form $\langle a, b, c \rangle$ or $[a, b, c]$ or
 a, b, c .

$$\therefore \frac{a}{l} = \frac{b}{m} = \frac{c}{n} = k \text{ (say)}$$

$$\Rightarrow a = kl, b = km, c = kn \quad \text{--- (1)}$$

where k is the constant of proportionality. Squaring & adding these eqns. we get

$$a^2 + b^2 + c^2 = k^2 (l^2 + m^2 + n^2)$$

$$\therefore a^2 + b^2 + c^2 = k^2 \quad [\because l^2 + m^2 + n^2 = 1]$$

$$\Rightarrow k = \pm \sqrt{a^2 + b^2 + c^2}$$

$$l = \frac{a}{\pm \sqrt{a^2 + b^2 + c^2}}, \quad m = \frac{b}{\pm \sqrt{a^2 + b^2 + c^2}}, \quad n = \frac{c}{\pm \sqrt{a^2 + b^2 + c^2}}$$

If a line has direction ratio $\langle -18, 12, -4 \rangle$, then determine its d.c's.

Direction cosines & direction ratios of the line segment joining 2 pts

Suppose $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ are two points

The direction ratios are $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

Angle betⁿ two lines =

① The angle betⁿ two lines having d.c's l_1, m_1, n_1 & l_2, m_2, n_2
 $\theta \rightarrow$ angle.

$$\cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

$$\Rightarrow \theta = \cos^{-1} (l_1 l_2 + m_1 m_2 + n_1 n_2)$$

② If $\langle a_1, b_1, c_1 \rangle$ and $\langle a_2, b_2, c_2 \rangle$ be d.c's of two lines then the

d.c's are $\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$

$$\Rightarrow \theta = \cos^{-1} \left(\frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} \right)$$

Ex-15

Find the acute angle betⁿ the lines whose dir's are

~~90~~

$(1, 1, 2)$ and $(\sqrt{3}-1, -\sqrt{3}-1, 4)$ resp.

$$(\sqrt{3}+1)^2 = 3+1+2\sqrt{3}$$

Solⁿ

Here $\langle a_1, b_1, c_1 \rangle = \langle 1, 1, 2 \rangle$ and

$\langle a_2, b_2, c_2 \rangle = \langle \sqrt{3}-1, -\sqrt{3}-1, 4 \rangle$

$$\text{Then } \cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{\sqrt{3}-1 + (-\sqrt{3}-1) + 8}{\sqrt{6} \times \sqrt{(\sqrt{3}-1)^2 + (-\sqrt{3}-1)^2 + 16}}$$

$$= \frac{6}{\sqrt{6} \times \sqrt{3+1-2\sqrt{3} + 3+1+2\sqrt{3}+16}} = \frac{6}{\sqrt{6} \times \sqrt{24}}$$

$$= \frac{6}{\sqrt{144}} = \frac{6}{12} = \frac{1}{2}$$

$$\therefore \cos \theta = \frac{1}{2} \Rightarrow \theta = \cos^{-1} \frac{1}{2} = 60^\circ$$

\therefore Hence the reqd. Angle $\theta = 60^\circ$

Case-1 condition for \perp^{th} :

$$\text{Here } \theta = 90^\circ$$

$$\cos \theta = \cos 90^\circ = 0$$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

In terms of dir's of the lines the \perp^{th} condⁿ is

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

Case-2

Condⁿ for \parallel^{th}

$$\text{Here } \theta = 0^\circ$$

$$\cos 0^\circ = 1$$

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = 1$$

For dir's

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = 1$$



AB \parallel CD

CD

Unit-4 Plane

Plane - A plane is defined as surface such that the st. line joining any two points on the surface lies on it.

Note $x=0$, is the eqⁿ of yz -plane
 $y=0$, is the eqⁿ of xz -plane
 $z=0$, is the eqⁿ of xy -plane.

Normal form - The eqⁿ of plane is

$$\boxed{lx + my + nz = p} \quad \left\{ \because l^2 + m^2 + n^2 = 1 \right\}$$

Th^m-3 Equation of the plane through making off intercepts a, b, c on the co-ordinate axes

is

$$\boxed{\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1}$$

Plane through a given point -

The eqⁿ of any plane through (x_1, y_1, z_1)

is $\boxed{a(x - x_1) + b(y - y_1) + c(z - z_1) = 0}$.

Th^m-4 Find the eqⁿ of the plane passing through three non-collinear points (x_1, y_1, z_1) , (x_2, y_2, z_2) and (x_3, y_3, z_3) .

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

This is the reqd. eqⁿ of the plane

Thm-5: To reduce the eqⁿ $ax+by+cz+d=0$ of the plane from general form to normal form.

Solⁿ: The given eqⁿ is
 $ax+by+cz+d=0$

Hence, in general $\langle a, b, c \rangle$ are the dir^s of the normal to the plane. Let $\langle l, m, n \rangle$ be the direction cosine of the normal so, we have

$$l = \frac{a}{\pm \sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\pm \sqrt{a^2+b^2+c^2}}$$

$$n = \frac{c}{\pm \sqrt{a^2+b^2+c^2}}$$

Plane passing through the intersection of two points:

Consider two intersecting planes given by the eq^s

$$a_1x + b_1y + c_1z + d_1 = 0 \quad \text{--- (1)}$$

$$a_2x + b_2y + c_2z + d_2 = 0 \quad \text{--- (2)}$$

We consider the eqⁿ

$$(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$$

where k is a parameter

$$\text{i.e. } (a_1 + ka_2)x + (b_1 + kb_2)y + (c_1 + kc_2)z + (d_1 + kd_2) = 0$$

which is also a plane.

— Angle between two planes =

— Angle between two planes is equal to the angle between their normals.

If the eqⁿ of two planes be

$$a_1x + b_1y + c_1z + d_1 = 0$$

$$\text{and } a_2x + b_2y + c_2z + d_2 = 0$$

If θ be the angle betⁿ the planes

$$\cos \theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(i) — the plane will be parallel if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

(ii) and perpendicular if $\cos \theta = 0$

$$\text{i.e. } a_1a_2 + b_1b_2 + c_1c_2 = 0$$

(iii) Two planes are identical if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$$

Note-1: — the distance of the point (x_0, y_0, z_0) from the plane $ax + by + cz + d = 0$ is given by

$$\textcircled{1} = \left| \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

Note-2

Distance betⁿ two parallel planes $ax + by + cz + d = 0$ and $ax + by + cz + d_1 = 0$ is given by

$$\textcircled{1} = \left| \frac{d - d_1}{\sqrt{a^2 + b^2 + c^2}} \right|$$

Examples

Ex-1 Find the eqⁿ of the plane which passes through $(4, -2, 1)$ and is perpendicular to the line whose direction ratios are $7, 2, -3$.

Solⁿ Equation of the plane through $(4, -2, 1)$ is

$$a(x-4) + b(y+2) + c(z-1) = 0 \quad \text{--- (i)}$$

Since the dir^s of the normal to the plane are $7, 2, -3$.

$$\therefore a = 7, b = 2, c = -3.$$

Putting these values of a, b and c in (i) we get

$$7(x-4) + 2(y+2) - 3(z-1) = 0$$

$$\text{or, } 7x + 2y - 3z - 21 = 0.$$

is the reqd. eqⁿ of the plane.

Ex-2 Find the eqⁿ of the plane which passes through the point $(1, -1, 4)$ and is parallel to the plane $2x - 3y + 7z = 11$.

Solⁿ Any plane parallel to the plane $2x - 3y + 7z - 11 = 0$ is of the form $2x - 3y + 7z + k = 0$ --- (1)

Since it passes through $(1, -1, 4)$

$$2(1) - 3(-1) + 7(4) + k = 0$$

$$\text{or } 2 + 3 + 28 + k = 0$$

$$\Rightarrow k = -33$$

put $k = -33$ in eqⁿ (1) we get

$$2x - 3y + 7z - 33 = 0$$

Ex-3 Find the eqⁿ of the plane containing the line of intersection of the planes $x+y+z+1=0$, $2x-3y+5z-2=0$ and passing through the point $(-1, 2, 1)$.

Solⁿ Eqⁿ of any plane passing through the line of intersection of the planes

$$x+y+z+1=0 \text{ and } 2x-3y+5z-2=0$$

and passing through $(-1, 2, 1)$.

$$\Rightarrow (x+y+z+1) + \lambda(2x-3y+5z-2) = 0 \quad \text{--- (i)}$$

Since (i) also passes through $(-1, 2, 1)$

$$\therefore (-1+2+1) + \lambda(-2-6+5-2) = 0$$

$$\text{or } 3 + \lambda(-3) = 0 \quad \text{or } \boxed{\lambda = \frac{3}{5}}$$

Putting this value of $\frac{3}{5}$ in (i) we get

$$(x+y+z+1) + \frac{3}{5}(2x-3y+5z-2) = 0$$

$$\text{or } 5x+5y+5z+5+6x-9y+15z-6=0$$

or $11x-4y+20z-1=0$ is the reqd. eqⁿ of the plane.

Ex-4 Find the eqⁿ of the plane passing through the point $(-1, -1, 2)$ and \perp to the planes

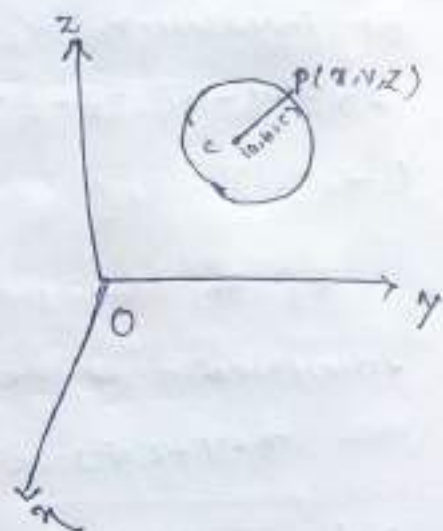
$$5x+2y-3z=1$$

$$\text{and } 5x-4y+z=5$$

Sphere

→ A sphere is the locus of a point in space which moves in such a way that it remains always at a constant distance from a fixed point.

→ The fixed point is called the centre and the constant distance is called radius of the sphere.



Theorem → The eqⁿ of the sphere with centre at the point (a, b, c) and radius r is given by

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

General eqⁿ of a sphere →

General eqⁿ of a sphere expressed in the form

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

→ Hence the centre of the sphere is $(-u, -v, -w)$ and radius of the sphere is $\sqrt{u^2 + v^2 + w^2 - d}$.

Note

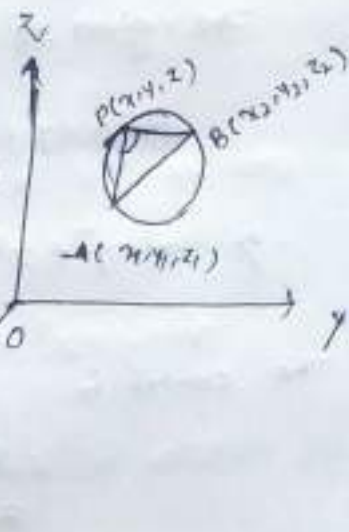
If $u^2 + v^2 + w^2 < d$ sphere may be called an imaginary sphere.

If $u^2 + v^2 + w^2 > d$ sphere may be called an real sphere.

Ex^m-2 The co-ordinates of end point of a diameter of a sphere are (x_1, y_1, z_1) and (x_2, y_2, z_2) .
Find the eqⁿ of the sphere.

Solⁿ Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two end points of diameter of a sphere.

using condition of perpendicularity



$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0$$

Ex^m-3 To find the eqⁿ of the sphere through four given points (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) and (x_4, y_4, z_4) .

Solⁿ Let the eqⁿ of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

since it passes through (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) and (x_4, y_4, z_4) is given by

$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0 \quad \text{--- (1)}$$

$$x_2^2 + y_2^2 + z_2^2 + 2ux_2 + 2vy_2 + 2wz_2 + d = 0 \quad \text{--- (2)}$$

$$x_3^2 + y_3^2 + z_3^2 + 2ux_3 + 2vy_3 + 2wz_3 + d = 0 \quad \text{--- (3)}$$

$$x_4^2 + y_4^2 + z_4^2 + 2ux_4 + 2vy_4 + 2wz_4 + d = 0 \quad \text{--- (4)}$$

Solving these eq^s we get the req^d sphere.

Ex-1 Find the centre and radius of the sphere

$$4x^2 + 4y^2 + 4z^2 - 16x - 24z + 3 = 0$$

Solⁿ The given eqⁿ of sphere is

$$4x^2 + 4y^2 + 4z^2 - 16x - 24z + 3 = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - 4x - 6z + 3/4 = 0$$

The centre is $2u = -4 \Rightarrow u = -2$

$$2v = 0 \Rightarrow v = 0$$

$$2w = -6 \Rightarrow w = -3$$

$$(u, v, w) = (-2, 0, -3)$$

$$\text{Radius } r = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{4 + 0 + 9 - \frac{3}{4}}$$

$$= \sqrt{13 - \frac{3}{4}} = \sqrt{\frac{49}{4}} = \frac{7}{2}$$

Hence the centre is $(-2, 0, -3)$ and radius is $\frac{7}{2}$.

Ex-2 Find the eqⁿ of the sphere on the joint of $(2, 3, 5)$ and $(4, 9, -3)$ as diameter?

Solⁿ We know that eqⁿ of a sphere whose end points of the diameter are (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

∴ Eqⁿ of the sphere whose end points of the diameter are $(2, 3, 5)$ and $(4, 9, -3)$ is

$$(x-2)(x-4) + (y-3)(y-9) + (z-5)(z+3) = 0$$

$$\rightarrow x^2 + y^2 + z^2 - 6x - 12y - 2z + 8 + 27 - 15 = 0$$

$$\rightarrow x^2 + y^2 + z^2 - 6x - 12y - 2z + 20 = 0$$

is the reqd. eqⁿ of the sphere.

Ex-3

find the eqⁿ of the sphere which passes through the points $(0, 0, 0)$, $(0, 1, 0)$, $(1, 0, 0)$ & $(0, 0, 1)$.

Solⁿ

Let the eqⁿ of sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{--- (1)}$$

Since $(0, 0, 0)$ lies on (i)

$$\therefore d = 0 \quad \text{--- (2)}$$

Again $(0, 1, 0)$ lies on (i)

$$\therefore 1 + 2v + d = 0$$

$$\text{or } 1 + 2v = 0 \Rightarrow v = -\frac{1}{2} \quad (\because d = 0)$$

Also $(1, 0, 0)$ lies on (i)

$$\therefore 1 + 2u + d = 0$$

$$\text{or } 1 + 2u = 0$$

$$\Rightarrow u = -\frac{1}{2}$$

→ Again $(0,0,1)$ lies on (i)

$$\therefore 1 + 2d + d = 0$$

$$\Rightarrow 1 + 2d = 0 \quad \Rightarrow \quad d = -\frac{1}{2} \quad (\because d=0)$$

put $u = v = w = -\frac{1}{2}$ and $d = 0$ in (i) we get

$$x^2 + y^2 + z^2 - x - y - z = 0$$

which is the reqd. eqⁿ of the sphere.

Ex-4 Find the eqⁿ of the sphere with centre $(3, -2, 5)$ and radius 4.

Solⁿ The reqd. eqⁿ of the sphere is

$$(x-3)^2 + (y+2)^2 + (z-5)^2 = 4^2 = 16$$

$$\Rightarrow x^2 + y^2 + z^2 - 6x + 4y - 10z + 22 = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - 6x + 4y - 10z + 22 = 0$$

Imp
Ex-5 Find the eqⁿ of the sphere which passes through the points $(0,0,0)$, $(-a, b, c)$, $(a, -b, c)$ and $(a, b, -c)$.