Differential Equations

PRAGYAN PRIYADARSINI
LECTURER IN MATHEMATICS
GOVT. POLYTECHNIC JAJPUR

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Definition

- An equation involving
  - independent variable,
  - dependent variable and
  - derivative of dependent variable with respective to the independent variable or variables
- is known as **DIFFERENTIAL EQUATION**.

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For example:

$$\frac{dy}{dx} + 3y^2 = 9x$$

- In the above equation:
  - x = independent variable
  - y = dependent variable
  - $\frac{dy}{dx}$ = derivative of dependent variable (i.e. 'y') with respective to the independent variable or variables (i.e. 'x')

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Types of Differential Equations

- Differential Equations are of 2 types:

A. Ordinary differential equations (O.D.E)

B. Partial differential equations (P.D.E)

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Ordinary differential equations (O.D.E)

- Differential equations involving derivatives w.r.t only one independent variable is called Ordinary differential equations (O.D.E)

Example:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 9x = 0$$

- Here the derivatives includes only one independent variable i.e. 'x'

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Partial differential equations (P.D.E)

- Differential equations involving derivatives w.r.t more than one independent variable is called Partial differential equations (P.D.E)

Example:

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = 5u$$

Here u = f (x ,y, z), therefore

- u → dependent variable
- x, y, z → independent variables

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Order of the Differential equation

- Order of the differential equation is the highest order of the derivatives occurring in it.
- As we already know:

$\frac{dy}{dx}$ ⟹ 1st order derivative

$\frac{d^2y}{dx^2}$ ⟹ 2nd order derivative

$\frac{d^3y}{dx^3}$ ⟹ 3rd order derivative

$\frac{d^ny}{dx^n}$ ⟹ nth order derivative

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Lets see few examples:

E.g. 1:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 9x = 0$$

- Order = 2

E.g. 2:

$$\left(\frac{dy}{dx}\right) + x^2 = \frac{d^3y}{dx^3}$$

- Order = 3

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Degree of the Differential equation

- Degree of the Differential equation is the highest power of the highest order derivative after the equation has been freed from radicals and fractions.

Lets see few examples:

E.g. 1:

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 = 9x$$

- Order = 3
- Degree = 1

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E.g. 2:

$$\frac{d^2y}{dx^2} = \sqrt{3 + \frac{dy}{dx}}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = 3 + \frac{dy}{dx}$$ [*squaring both sides*]

- Order = 2
- Degree = 2

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E.g. 3:

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2} = 3\left(\frac{d^2y}{dx^2}\right)$$

[squaring both sides]

$$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^{5} = \left\{3\left(\frac{d^2y}{dx^2}\right)\right\}^2$$

$$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^{5} = 9\left(\frac{d^2y}{dx^2}\right)^2$$

- Order = 2
- Degree = 2

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Solution of Differential equation

- Let us take a differential eq^n and a function

$$\frac{d^2y}{dx^2} + y = 0 \quad \text{(1)}$$

$$y = a \sin (x+b) \quad \text{(2)}$$

[*where a, b are real number*]

then

$$\Rightarrow \frac{dy}{dx} = a \cos(x+b)$$ [*differentiating eq^n* (2)]

$$\Rightarrow \frac{d^2y}{dx^2} = - a \sin (x+b)$$ [*differentiating again*]

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now putting the values of y & $\frac{d^2y}{dx^2}$ in eq^n (1)

L.H.S ➡ $\frac{d^2y}{dx^2} + y = -a\sin(x+b) + a\sin(x+b) = 0$

R.H.S ➡ 0

L.H.S = R.H.S

- so we conclude that:

$y = a\sin(x+b)$ is solution of differential equation

$\frac{d^2y}{dx^2} + y = 0$ as it satisfies the equation.

Note:- a function is said to be solution of a differential equation if it satisfies the equation.

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Two types of solution

A. General or complete solution

B. Particular solution

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General or complete solution

- A solution which contains the number of arbitrary constant equal to the order of the differential equation is called a general solution.

Example:

$y = a \sin(x+b)$ is general solution of differential equation $\frac{d^2y}{dx^2} + y = 0$

- Order of differential equation = 2
- **a**, **b** are two arbitrary constants in the solution.

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Particular solution

- A particular solution of a differential equation is a solution obtained from the general solution by giving some particular values to the arbitrary constants.

Example:

$y = 2\sin(x+5)$ is particular solution of differential equation $\frac{d^2y}{dx^2} + y = 0$

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Solution of Differential equation

Solution of 1^{st} order and 1^{st} degree equation by:

A. **Separation of variables**

B. **Solution of linear Differential equation of first order**

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Separation of variables

- Consider the Differential equation

$$\frac{dy}{dx} = f(x,y) \quad (1)$$

- Equation (1) can be separable of variables

$$\Rightarrow \frac{dy}{dx} = f_1(x)\, f_2(y)$$

$$\Rightarrow \frac{dy}{f_2(y)} = f_1(x)\, dx$$

- Integrating both sides

$$\Rightarrow \int \frac{dy}{f_2(y)} = \int f_1(x)\, dx + C$$

- Which is a complete solution

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Question 1

- Solve $$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

- Sol^n

[*cross-multiplying*]

$$\Rightarrow \frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

[*integrating both sides*]

$$\Rightarrow \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$$\Rightarrow \tan^{-1} y = \tan^{-1} x + C$$ answer

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Question 2

- Solve

$$e^x \tan y\, dx + (1+e^x) \sec^2 y\, dy = 0$$

- Sol[n]

$\Rightarrow$ $e^x \tan y\, dx + (1+e^x) \sec^2 y\, dy = 0$

$\Rightarrow$ $(1+e^x) \sec^2 y\, dy = -e^x \tan y\, dx$

$\Rightarrow$ $\dfrac{\sec^2 y\, dy}{\tan y} = \dfrac{-e^x\, dx}{(1+e^x)}$

[integrating both sides]

$\Rightarrow$ $\displaystyle\int \frac{\sec^2 y\, dy}{\tan y} = \int \frac{-e^x\, dx}{(1+e^x)}$ ——— (1)

I_1 $\qquad$ I_2

contd..

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- For I_1

Let $\tan y = u$

$\Rightarrow \sec^2 y = \dfrac{du}{dy}$

$\Rightarrow \sec^2 y \, dy = du$

$\Rightarrow \displaystyle\int \frac{\sec^2 y \, dy}{\tan y} = \int \frac{du}{u}$

$\Rightarrow = \log u$

$\Rightarrow = \log \tan y$

- For I_2

Let $(1+ e^x) = v$

$\Rightarrow e^x = \dfrac{dv}{dx}$

$\Rightarrow e^x \, dx = dv$

$\Rightarrow -\displaystyle\int \frac{e^x \, dx}{(1+ e^x)} = -\int \frac{dv}{v}$

$\Rightarrow = -\log v$

$\Rightarrow = -\log (1+ e^x)$

Eg^n (1) becomes:

$\Rightarrow \log \tan y = -\log (1+ e^x) + C$ answer

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Solution of linear Differential equation of first order

- A differential equation in which the dependent variable and all its derivatives occur in the 1st degree only and are not multiplied together is called a Linear Differential equation.
- **Standard form** of linear differential equation (1st order)

$$\frac{dy}{dx} + Py = Q$$

- where P and Q may be constant or only a function of x.
- coefficient of $\frac{dy}{dx}$ is always unity.

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method of solution

- Step 1
  - Find I.F (Integrating factor)

$$\Rightarrow \quad e^{\int p\, dx}$$

- Step 2
  - Then the complete solution is given by

$$y \times I.F = \int \{Q \times (I.F)\}\, dx + C$$

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Question 1

- Solve $\dfrac{dy}{dx} + y\,\tan x = \sec x$

- Sol^n It is in its standard form

$\Rightarrow \dfrac{dy}{dx} + Py = Q$ [P = tan**x**] [Q = sec**x**]

I.F (Integrating factor) $\Rightarrow e^{\int p\,dx}$

$\Rightarrow e^{\int \tan x\,dx}$

$\Rightarrow e^{\log|\sec x|}$

$\Rightarrow \sec x$

contd..

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complete solution is given by:

$$y \times I.F = \int\{Q \times (I.F)\}\, dx + C$$

$$\Rightarrow y \times \sec\mathbf{x} = \int\{\sec\mathbf{x} \times \sec\mathbf{x}\}\, dx + C$$

$$\Rightarrow y \sec\mathbf{x} = \int\{\sec^2\mathbf{x}\}\, dx + C$$

$$\Rightarrow y \sec\mathbf{x} = \tan\mathbf{x} + C$$

answer

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Question 2

- Solve $$x\frac{dy}{dx} + 2y = 4x^2$$

- Sol^n

it is not in its standard form

$$\Rightarrow \frac{dy}{dx} + \frac{2y}{x} = 4x$$

[*divide by 'x' on both sides*]

now it is in the standard form

$[P = \frac{2}{x}]$

$[Q = 4x]$

I.F $\Rightarrow e^{\int p\,dx}$

$$\Rightarrow e^{\int \frac{2}{x} dx} \Rightarrow e^{2\int \frac{1}{x} dx}$$

$$\Rightarrow e^{2\log x} \Rightarrow e^{\log x^2}$$

$$\Rightarrow x^2$$

contd..

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complete solution is given by:

$$y \times I.F = \int \{Q \times (I.F)\}\, dx + C$$

$\Rightarrow$ $y\,x^2 = \int (4x \,.\, x^2)\, dx + C$

$\Rightarrow$ $y\,x^2 = \int (4x^3)\, dx + C$

$\Rightarrow$ $y\,x^2 = \dfrac{4x^4}{4} + C$

$\Rightarrow$ $y\,x^2 = x^4 + C$ answer

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Question 3

- Solve $$(1+x^2)\frac{dy}{dx}+2xy - x^3 = 0$$

- Sol^n

it is not in its standard form

$$\Rightarrow \frac{dy}{dx} + \frac{2x\,y}{1+x^2} - \frac{x^3}{1+x^2} = 0$$ [*divide by '1+x²' on both sides*]

$$\Rightarrow \frac{dy}{dx} + \frac{2x\,y}{1+x^2} = \frac{x^3}{1+x^2}$$

$[\,P = \frac{2x}{1+x^2}\,]$

$[\,Q = \frac{x^3}{1+x^2}\,]$

now it is in the standard form

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I.F → $e^{\int p\,dx}$

⇒ $e^{\int \frac{2x}{1+x^2}\,dx}$

⇒ $e^{\int \frac{1}{t}\,dt}$

⇒ $e^{\log t}$

⇒ $t = 1+x^2$

[*let* $1+x^2 = t$]

[$2x = \frac{dt}{dx}$]

[$2x\,dx = dt$]

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contd..

complete solution is given by:

$$y \times I.F = \int\{Q \times (I.F)\}\,dx + C$$

$$\Rightarrow\quad y\,(1+x^2) = \int\left(\frac{x^3}{1+x^2}\right)(1+x^2)\,dx + C$$

$$\Rightarrow\quad y\,(1+x^2) = \int x^3\,dx + C$$

$$\Rightarrow\quad y\,(1+x^2) = \frac{x^4}{4} + C$$

answer

Vectors

Prepared by: **Pragyan Priyadarsini**
Lecturer in Mathematics
Jajpur Govt. Polytechnic, Odisha

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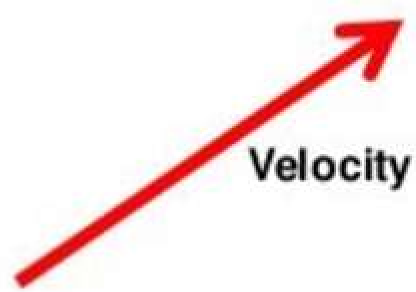
Scalars and Vectors

A scalar quantity is a quantity that has only magnitude.

A vector quantity is a quantity that has both a magnitude and a direction.

Scalar quantities
Length, Area, Volume,
Speed,
Mass, Density
Temperature, Pressure
Energy, Entropy
Work, Power

Vector quantities
Displacement, Direction,
Velocity, Acceleration,
Momentum, Force,
Electric field, Magnetic field



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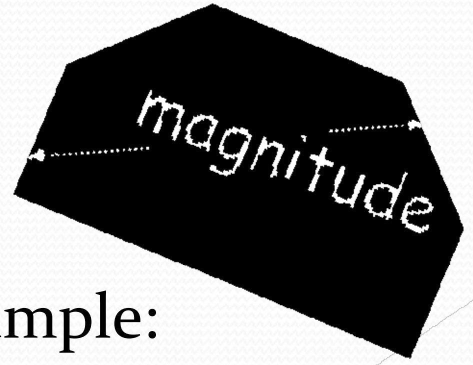
scalar

- only magnitude (size)
- 3.044, −7 and 2½

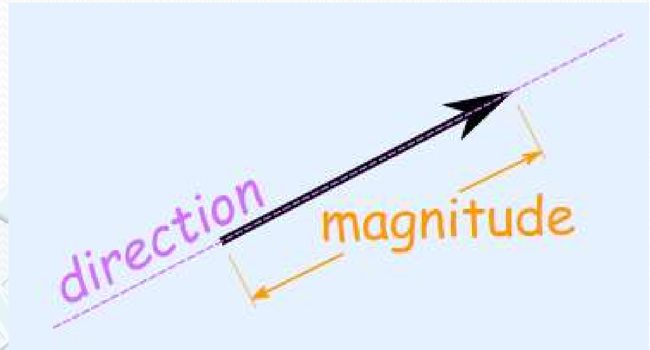


Example:

- Distance = 3 km
- Speed = 9 km/h
  (kilometers per hour)

vector

- magnitude and direction



- Displacement = 3 km
  Southeast
- Velocity = 9 km/h
  Westwards

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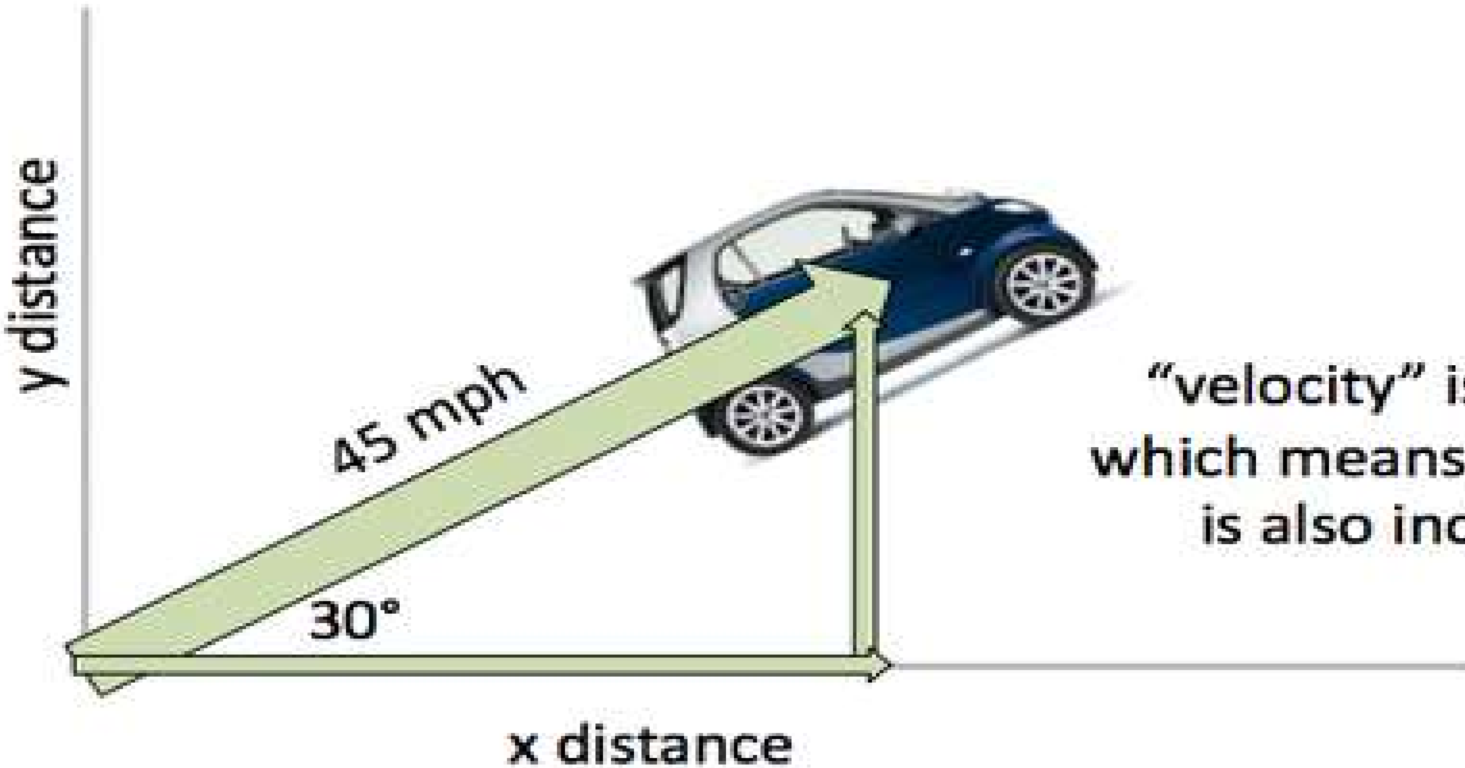
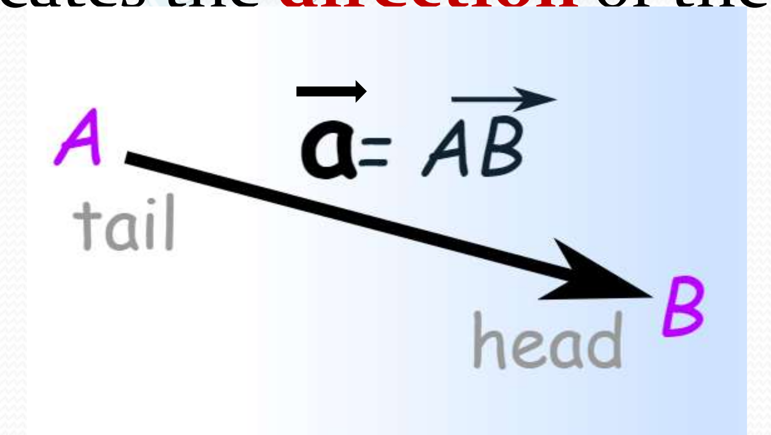
Distance is a scalar quantity, whereas displacement is a vector quantity.

Scalar and Vector Quantities



Vector - Notation/ Denoted as

- It is denoted as 'vector $\overrightarrow{AB}$' or 'vector $\vec{a}$'.
- point A from where the vector starts is called its initial point
- point B where it ends is called its terminal point.
- The distance between initial and terminal points of a vector is called the **magnitude** (or length) of the vector, denoted as $|\overrightarrow{AB}|$, or $|\vec{a}|$, or a.
- The arrow indicates the **direction** of the vector.



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Types of vector

- zero or null vector
- unit vector
- negative of a vector
- co-initial vectors
- co-terminus vectors
- equal vectors
- collinear or parallel vectors

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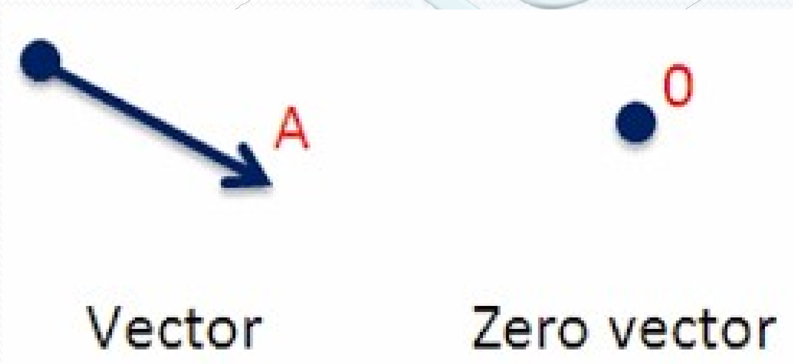
zero or null vector

- initial and terminal points coincident
- denoted by ➡ $\vec{0}$
- Magnitude ➡ 0 (zero)



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unit vector

- Magnitude ➡ **1** (unit magnitude, A= 1)
- denoted as ➡ $\hat{a}$
- purpose ➡ specify a **direction** in space

VECTOR A ← $\vec{A} = A\hat{A}$

A = magnitude of $\vec{A}$

$\hat{A}$ = unit vector along $\vec{A}$

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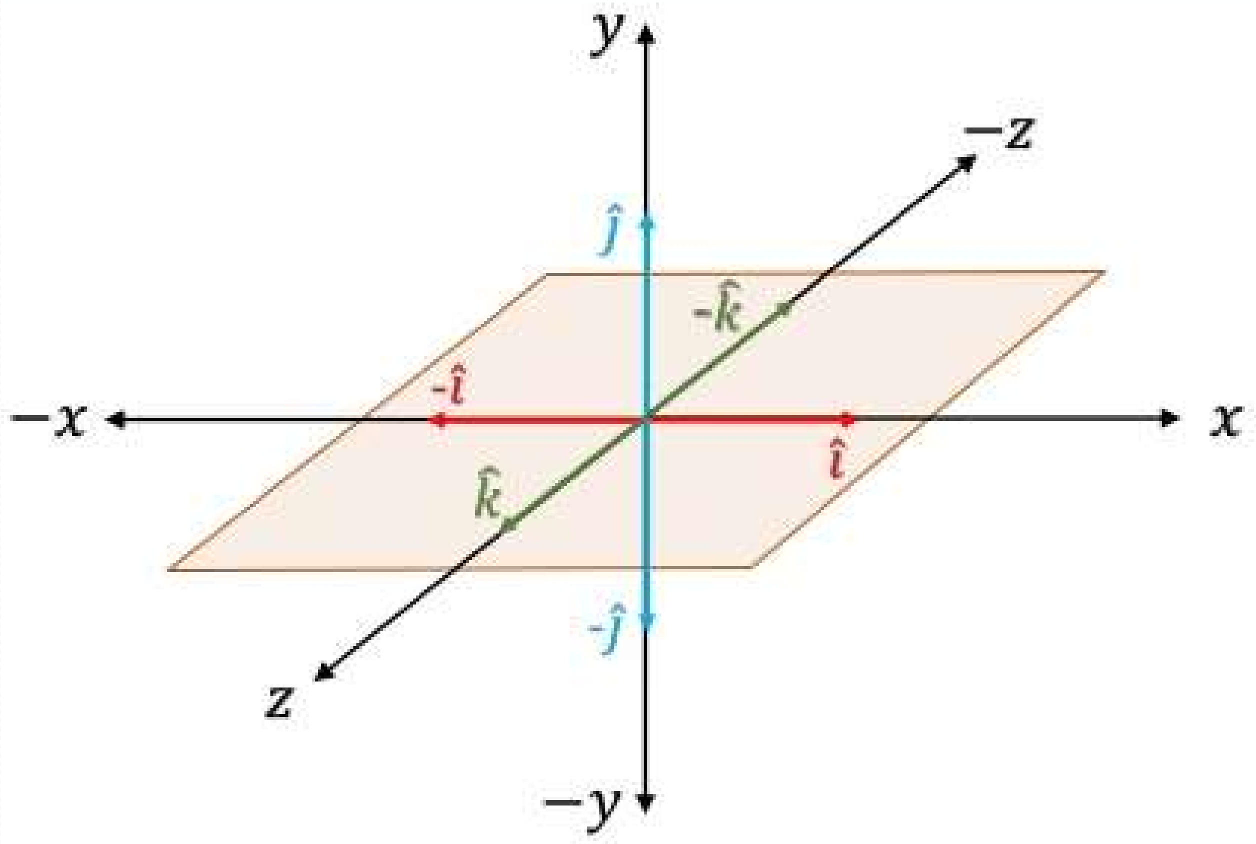
Cartesian unit vectors



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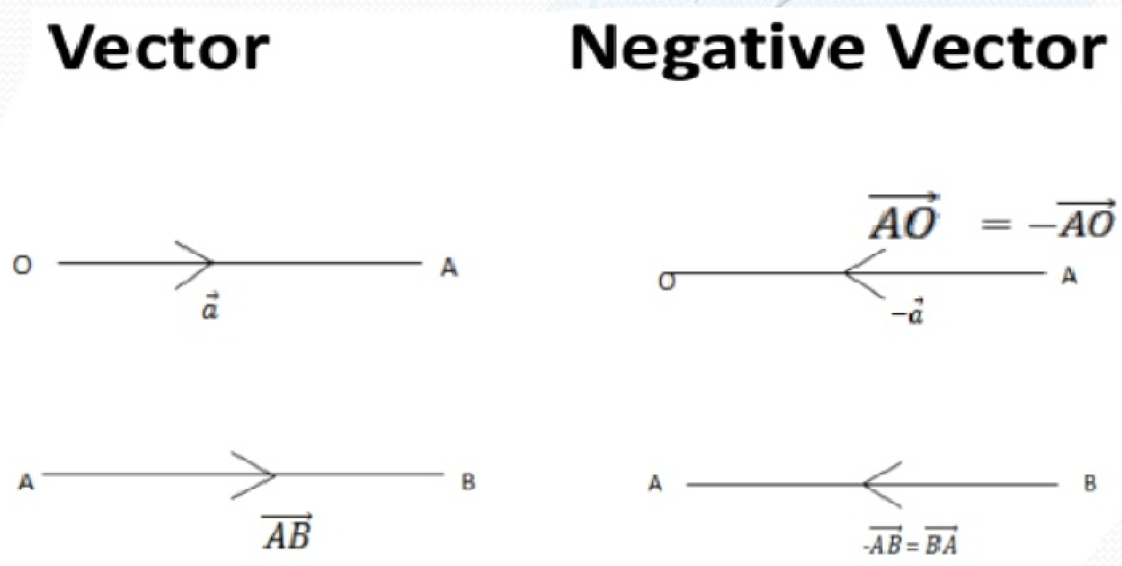
negative of a vector

- Vector of same magnitude
- but opposite direction



The negative vector of $\overrightarrow{AB}$ is - $\overrightarrow{AB} = \overrightarrow{BA}$

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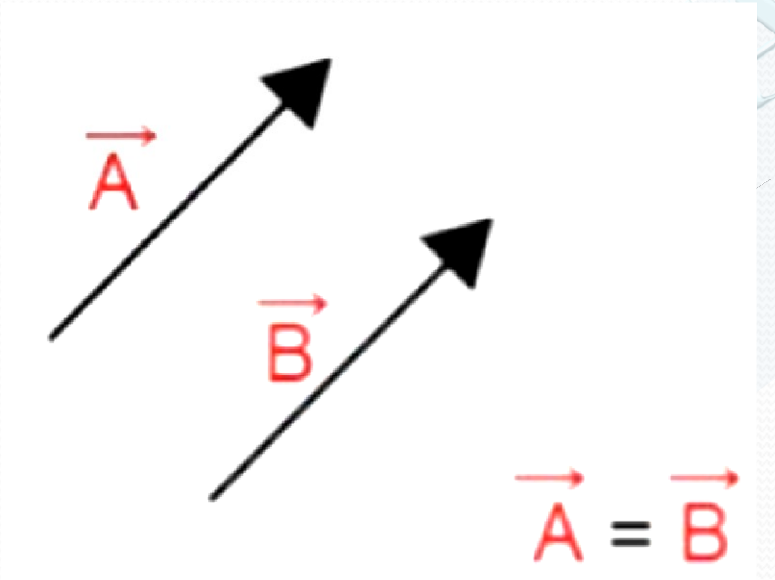
equal vectors

- same magnitude (size) as well as direction



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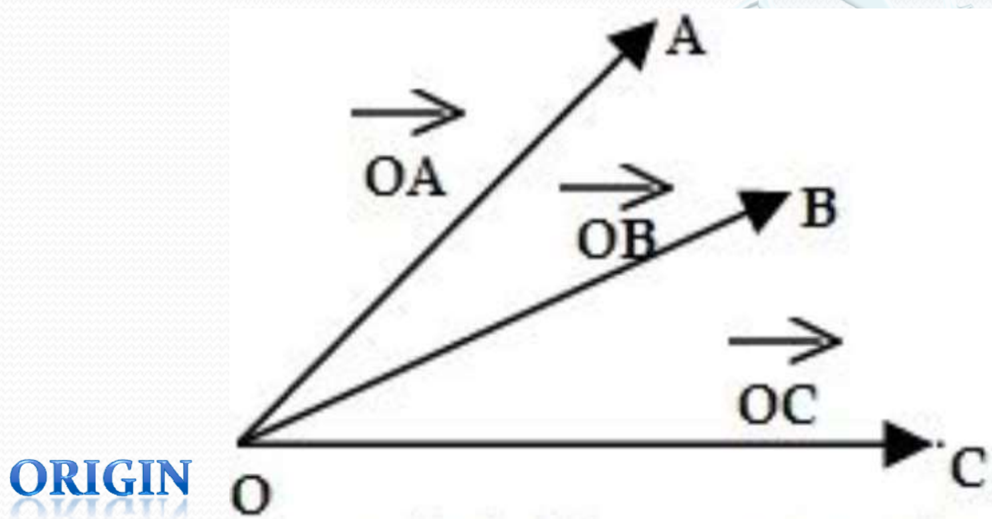
co-initial vectors

- same starting point



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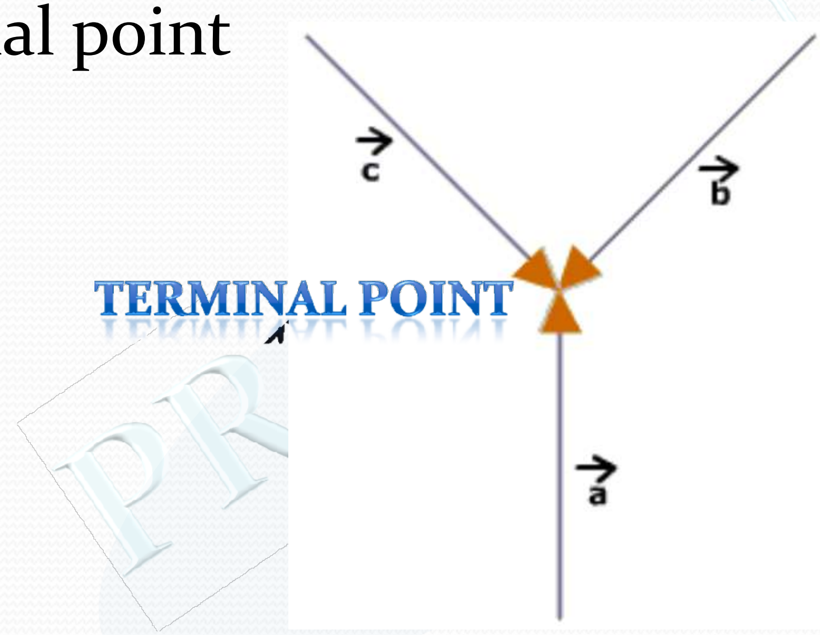
co-terminus vectors

- same terminal point



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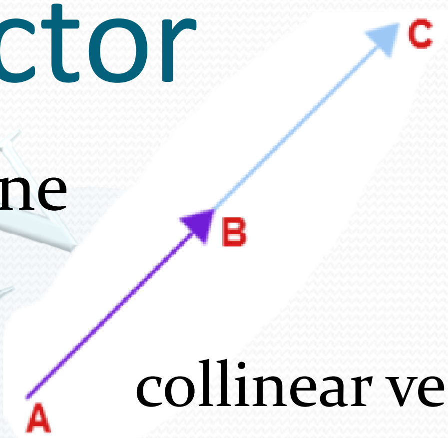
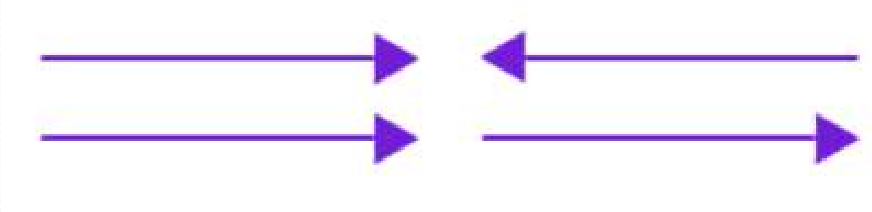
collinear or parallel vector

- **collinear vectors** ➡ lying on one line



collinear vector

- **parallel vectors** ➡ lying parallel to each other



parallel vector

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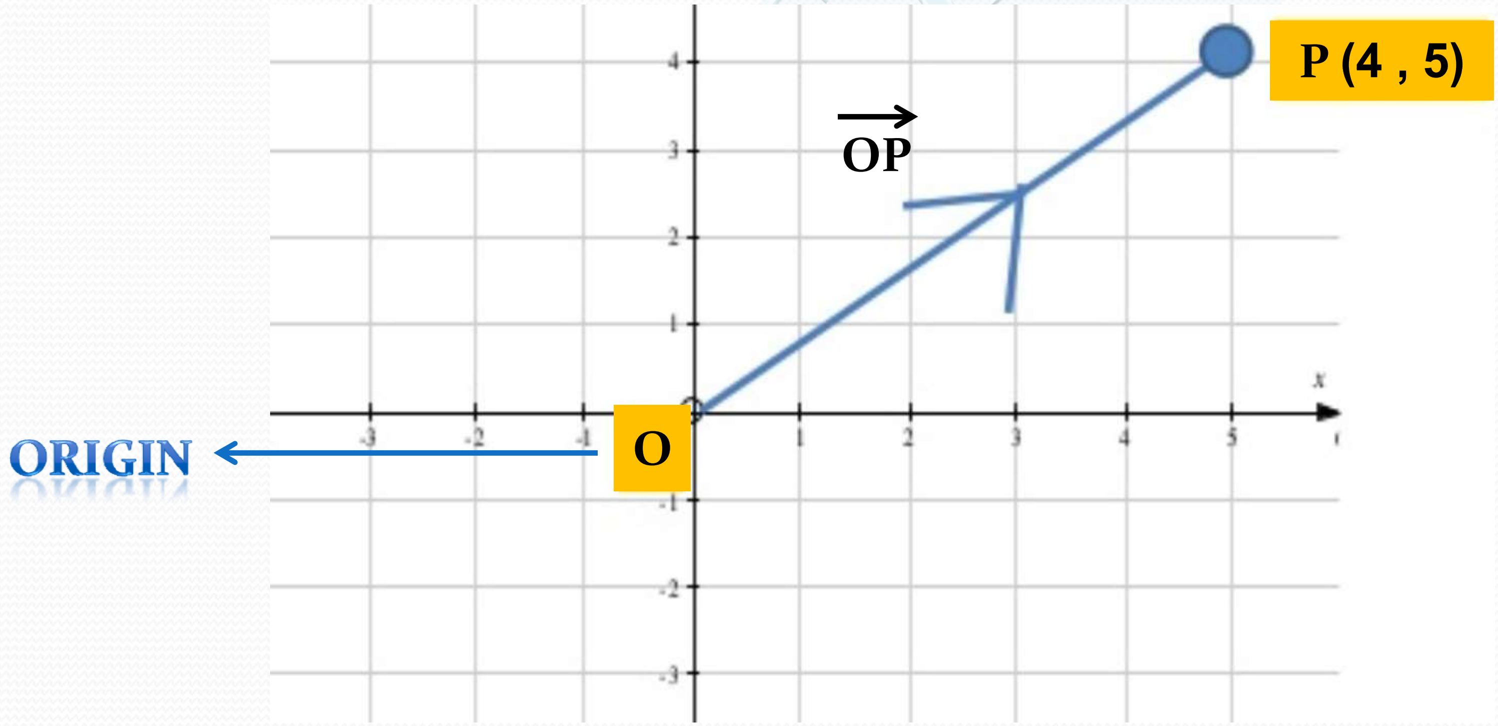
position vector

- Vector having initial point is at origin. Here $\overrightarrow{OP}$ is the position vector of point ‘P’.



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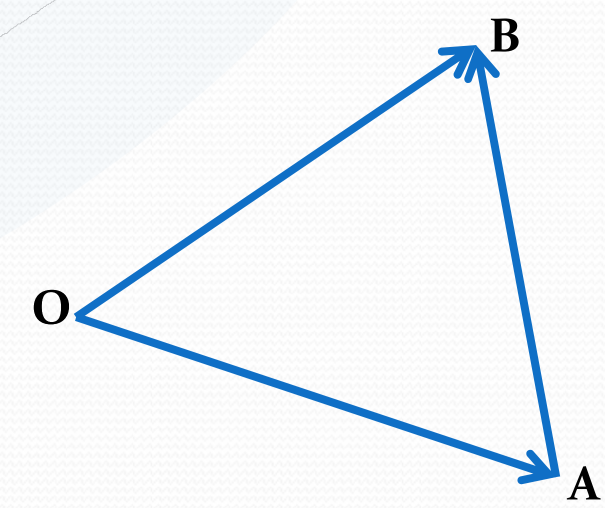
Representation of vectors in terms of the position vectors

- Let A and B be two given points.
- Then $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the position vectors of A and B
- Then $\overrightarrow{AB}$ can be represented as:

⇒ $\overrightarrow{AB}$ = p.v. of B – p.v. of A

⇒ $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$



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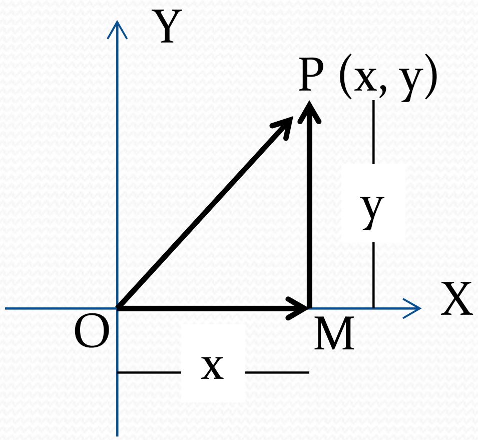
Components of a vector in two dimensions



Let $\hat{i}$ and $\hat{j}$ be the unit vectors along x-axis and y-axis

Then $\overrightarrow{OM} = x\hat{i}$

$\overrightarrow{MP} = y\hat{j}$

Then $\overrightarrow{OP} = x\,\hat{i} + y\,\hat{j}$ [*by Triangle law of addition*]

$|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$

as in Δ OPM

$(OP)^2 = (OM)^2 + (PM)^2$

$(OP)^2 = x^2 + y^2$

$OP = \sqrt{x^2 + y^2}$

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Components of a vector in three dimensions

Let P (x, y, z) be a point in 3D

Here $\hat{i}$, $\hat{j}$ & $\hat{k}$ are unit vectors along X-axis, Y-axis & Z-axis respectively

Then

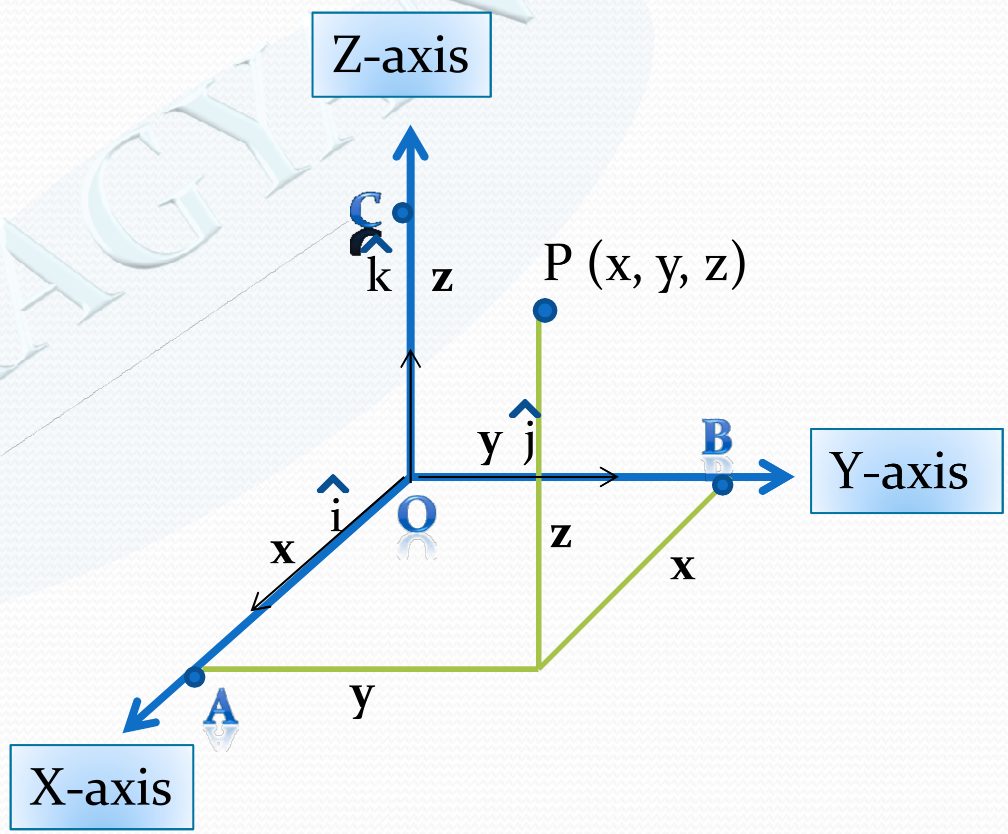
$\overrightarrow{OA} = x\,\hat{i}$

$\overrightarrow{OB} = y\,\hat{j}$

$\overrightarrow{OC} = z\,\hat{k}$

So $OP = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

and $|OP| = \sqrt{x^2 + y^2 + z^2}$



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Operations on vectors

- Addition of two vectors
  - Triangle law of addition
  - Parallelogram law of addition
- Subtraction of two vectors
- Multiplication
  - of a vector with a scalar
  - of two vectors by Dot product
  - of two vectors Cross product

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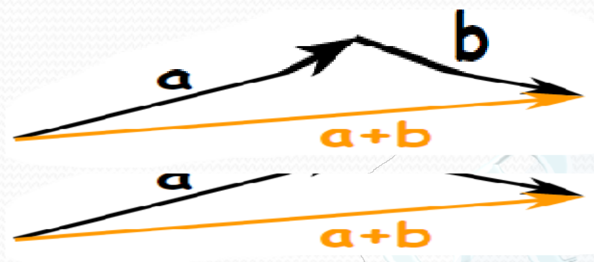
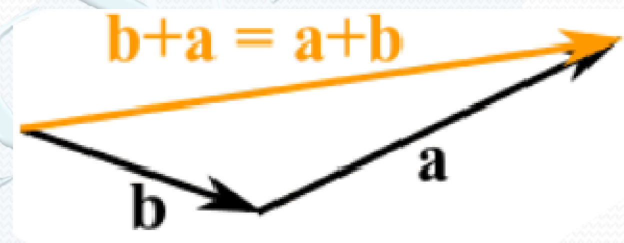
Adding Vectors by triangle law of addition

- We can add two vectors by joining them **head-to-tail**





triangle law of vector addition – states that if two vectors represented by 2 sides of the triangle then their sum is represented by the third side of the triangle but in the reverse order.

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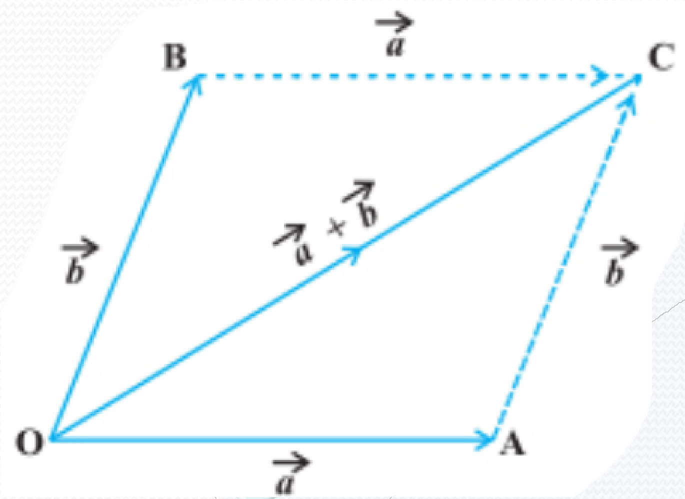
Adding Vectors by parallelogram law of vectors

- We can also add two vectors having a **same origin**



parallelogram law of vector addition – states that if 2 vectors $\vec{a}$ & $\vec{b}$ are represented by 2 adjacent sides of a parallelogram, then their sum $\vec{a} + \vec{b}$ is represented by the diagonal of the paralleogram through their initial point.

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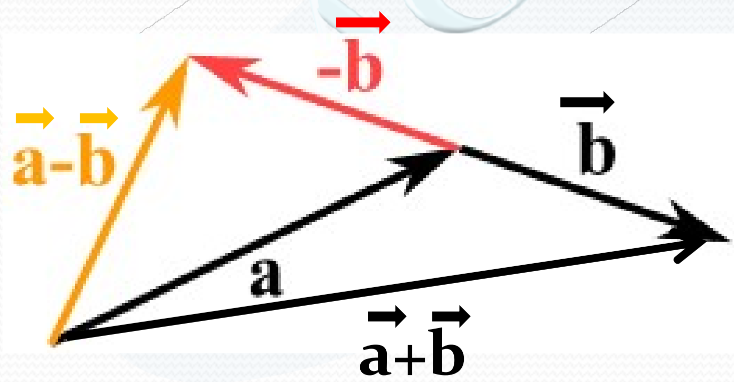
Subtracting vectors

- Let $\vec{a}$ and $\vec{b}$ be two vectors, reverse the direction of the vector $\vec{b}$ then add as usual:



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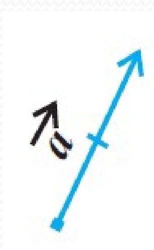
Multiplying a Vector by a Scalar

- product of the vector $\vec{a}$ by the scalar $\lambda = \lambda\vec{a}$

- magnitude $\Rightarrow$ $|\lambda\vec{a}| = |\lambda||\vec{a}|$

Example: $\vec{a} \times 2 = 2\vec{a}$

magnitude = $|2\vec{a}| = |2||\vec{a}|$ = 2a



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Addition of two vectors in components

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then $\overrightarrow{a + b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$\Rightarrow$ $(a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

Subtraction of two vectors in components

Then $\overrightarrow{a - b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) - (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$\Rightarrow$ $(a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$

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Multiplication of a vector with scalar

Let λ be a scalar

$$a = a_1 i + a_2 j + a_3 k$$

Then $\lambda \vec{a} = \lambda (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k})$

$$\lambda a_1 \hat{i} + \lambda a_2 \hat{j} + \lambda a_3 \hat{k}$$

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Multiplication of 2 vectors

- By using Scalar/ Dot product
- By using Vector/ Cross product

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Scalar or Dot Product

- Let $\vec{a}$ & $\vec{b}$ be two vectors.
- Then dot product of them is denoted by $\vec{\mathbf{a}} \, . \, \vec{\mathbf{b}}$
- and defined as:

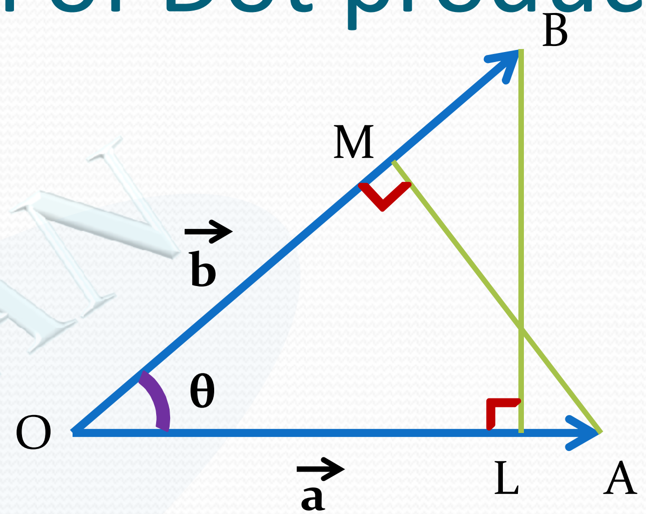
$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}| \times |\vec{\mathbf{b}}| \times \cos(\theta)$$

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \mathbf{a} \times \mathbf{b} \times \cos(\theta)$$

or $$\cos(\theta) = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}| \; |\vec{\mathbf{b}}|}$$

Video links

Geometrical representation of Dot product



Here in the given figure
θ is the angle between the vectors $\vec{a}$ & $\vec{b}$

Consider the right angled triangle ΔOBL
then

$$\cos\theta = \frac{b}{h} = \frac{OL}{OB} = \frac{OL}{|\vec{b}|}$$

$$|\vec{b}|\cos\theta = OL$$

and OL is known as projection of $\vec{b}$ on $\vec{a}$

as we know $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,OL$$

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = OL$$

So scalar projection of $\vec{b}$ on $\vec{a}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

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Video links

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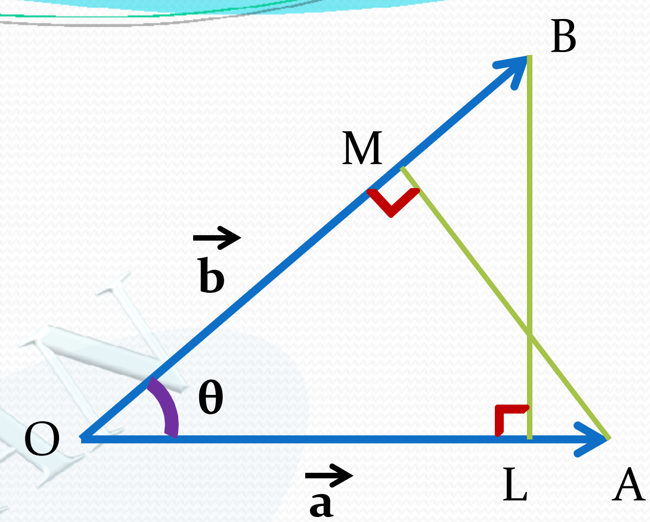

Again consider the right angled triangle ΔOAM then

$$\cos\theta = \frac{b}{h} = \frac{OM}{OA} = \frac{OM}{|\vec{a}|}$$

$$|\vec{a}|\cos\theta = OM$$

and OM is projection of $\vec{a}$ on $\vec{b}$

as we know $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

$$\vec{a}\cdot\vec{b} = |\vec{b}|\,OM$$

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = OM$$

So scalar projection of $\vec{a}$ on $\vec{b}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

Video links

Dot product in terms of components

Let

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

We have

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$$

$$\text{or } \hat{j}\cdot\hat{i} = \hat{k}\cdot\hat{j} = \hat{i}\cdot\hat{k} = 0$$

1

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$$

2

Then

$$\vec{a}\cdot\vec{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)\cdot\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$$

1

$$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

2

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} \Rightarrow \cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

Video links

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3. If $\vec{a}$ is perpendicular to $\vec{b}$

Then $\theta = 90^o \Rightarrow \cos\theta = \cos 90^o = 0$

$$\Rightarrow \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\Rightarrow 0 = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\Rightarrow 0 = \vec{a}\cdot\vec{b}$$

So $\vec{a} \perp \vec{b} \Rightarrow \vec{a}\cdot\vec{b} = 0$

Video links

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4 If $\vec{a}$ & $\vec{b}$ are parallel to each other

$$\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

5 $$\vec{a} \cdot \vec{a} = |\vec{a}|\,|\vec{a}| \cos 0$$

$$\Rightarrow |\vec{a}|^2 \cdot 1$$

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

Video links

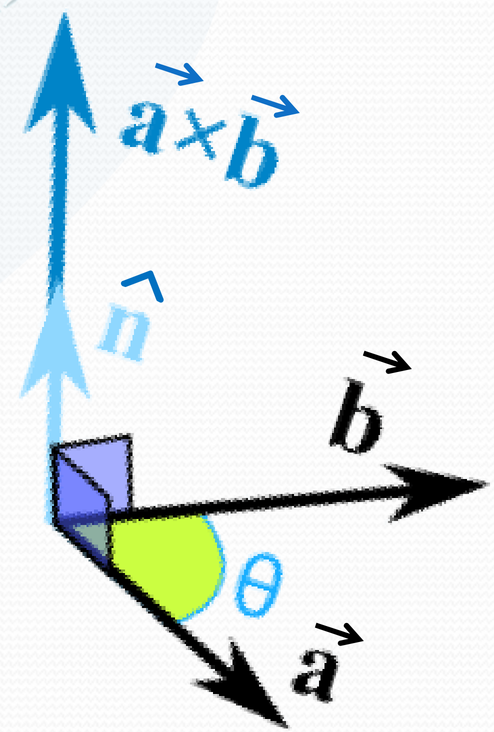
Vector or Cross Product

- The Vector Product of two vectors is denoted by $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ and defined as:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = |\vec{\mathbf{a}}|\ |\vec{\mathbf{b}}| \sin\theta \ .\ \hat{\mathbf{n}}$$

where:

$|\vec{a}|$ & $|\vec{b}|$ = magnitude
θ = angle between $\vec{a}$ & $\vec{b}$
$\hat{n}$ = unit vector perpendicular to both $\vec{a}$ & $\vec{b}$



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we have

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta \,.\, \hat{n}$$

Note:- 1

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta \,.\, |\hat{n}|$$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta \qquad \left(|\hat{n}| = 1\right)$$

$$\Rightarrow \frac{\vec{a} \times \vec{b}}{|\vec{a}|\,|\vec{b}|} = \sin\theta$$

we have

Note:- 2

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta \,.\, \hat{n}$$

$$\Rightarrow \vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|} \hat{n} \qquad \left(\text{putting formula of } \sin\theta\right)$$

$$\Rightarrow \vec{a} \times \vec{b} = |\vec{a} \times \vec{b}|\ \hat{n} \quad \Rightarrow \quad \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \hat{n}$$

So a unit vector $\hat{n}$ perpendicular to both $\vec{a}$ & $\vec{b}$ is given by

Video links

Continued..

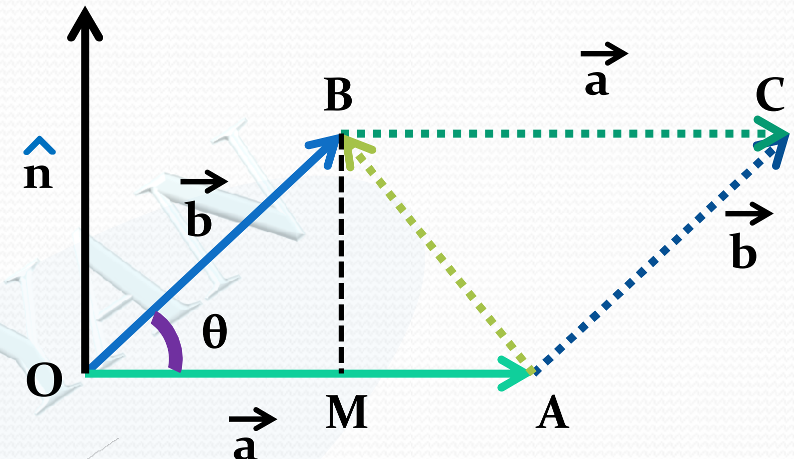
Geometrical representation of vector product

$\vec{OA} = \vec{a}$

$\vec{OB} = \vec{b}$

$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta \;\hat{n}$

⇒ $\vec{a} \times \vec{b} = |\vec{a}|\,(|\vec{b}| \sin\theta)\;\hat{n}$



$\sin\theta = \frac{p}{h} = \frac{BM}{OB} = \frac{BM}{|\vec{b}|}$

$|\vec{b}| \sin\theta = BM$

⇒ $\vec{a} \times \vec{b} = |\vec{a}|\; BM \;\hat{n}$

⇒ $|\vec{a} \times \vec{b}| = |\vec{a}|\,|BM|$

$=$ area of a parallelogram with sides $\vec{a}$ & $\vec{b}$

area of a parallelogram = base × height

Then it is concluded that:

Area of Δ ABC = ½ $|\vec{a} \times \vec{b}|$

Video links

Vector product in terms of components

Let

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

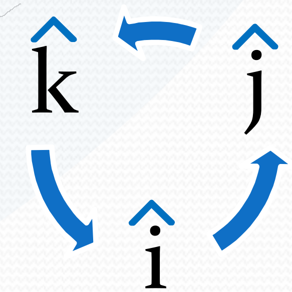
And from a right handed system of mutually perpendicular vector

We have:

$$\hat{i} \times \hat{j} = \hat{k} \quad \text{or} \quad \hat{j} \times \hat{i} = -\hat{k}$$

$$\hat{j} \times \hat{k} = \hat{i} \qquad \hat{k} \times \hat{j} = -\hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j} \qquad \hat{i} \times \hat{k} = -\hat{j}$$



And

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

So

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

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